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A Table of Integrals of the Exponential Integral*

Murray Geller and Edward W. Ng****

(March 11, 1969)

This is a compendium of indefinite and definite integrals of products of the Exponential Integral with elementary or transcendental functions. A substantial portion of the results are new.

Key words: Diffusion theory; exponential integral; indefinite integrals; quantum mechanics; radiative equilibrium; special functions; transport problems.

1. Introduction

Integrals of the exponential integral occur in a wide variety of applications. Examples of applications can be cited from diffusion theory [12],[1] transport problems [12], the study of the radiative equilibrium of stellar atmospheres [9], and in the evaluation of exchange integrals occurring in quantum mechanics [11]. This paper is an attempt to give an up-to-date exhaustive tabulation of such integrals.

All formulas for indefinite integrals in section 4 were derived from integration by parts and checked by differentiation of the resulting expressions. The formulas given in [1, 4, 5, 6, 7, 10, 14, and 15] have all been checked and included, with the omission of trivial duplications. Additional formulas were obtained either from the various integral representations, from the hypergeometric series for the exponential integrals, from multiple integrals involving elementary functions, from the existing literature [2, 3, 12, and 13], or by specialization of parameters of integrals over confluent hypergeometric functions [4] and [6].

Throughout this paper, we have adhered to the notations used in the NBS Handbook [8] and we have also assumed the reader's familiarity with the properties of the exponential integral. In addition, the reader should also attend to the following conventions:

(i) The integration constants have been omitted for the indefinite integrals;
(ii) the parameters a, b, and c are real and positive except where otherwise stated;
(iii) unless otherwise specified, the parameters n and k represent the integers 0, 1, 2 . . ., whereas the parameters p, q, and μ and ν may be nonintegral;
(iv) the integration symbol $\oint$ denotes a Cauchy principal value;
(v) x, y, and t represent real variables.

2. Glossary of Functions and Notations

ber (x), bei (x)	Thomson functions	
$B_x(p, q)$	Incomplete beta function	$\int_0^x t^{p-1}(1-t)^{q-1}dt$
$Ci(x)$	Cosine integral	$-\int_x^\infty \frac{\cos t}{t}\,dt$
$e_n(x)$	Truncated exponential	$\sum_{m=0}^{n} \frac{x^m}{m!}$

*An invited paper. This paper presents the results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract No. NAS7-100 sponsored by the National Aeronautics and Space Administration.

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[1] Figures in brackets indicate the literature references at the end of this paper.

$E_1(x)$	Exponential integral	$\int_x^\infty \frac{e^{-t}}{t}\,dt$
$Ei(x)$	Exponential integral	$-\int_{-x}^\infty \frac{e^{-t}}{t}\,dt$
erf (x)	Error function	$\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$
erfc (x)	Error function	$\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$
${}_1F_1(a;\, b;\, x)$	Confluent hypergeometric function	$\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\cdot\frac{x^n}{n!}$
${}_2F_1(a,\, b;\, c;\, x)$	Hypergeometric function	$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\cdot\frac{x^n}{n!}$
${}_pF_q$	Generalized hypergeometric function	
$\mathbf{H}_p(x)$	Struve function	
$I_p(x)$	Bessel function of imaginary argument	
$J_p(x)$	Bessel function	
$K_p(x)$	Bessel function of imaginary argument	
$li(x)$	Logarithmic integral	$\int_0^x \frac{dt}{\ln t},\ x>1$
$L_2(x)$	Euler's dilogarithm	$-\int_0^x \frac{\ln(1-t)}{t}\,dt$
$L_n(x)$	Laguerre polynomial	$\frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x})$
$M_{p,\,q}(x)$	Whittaker function	
$\binom{m}{n}$	Binomial coefficient	$\frac{m!}{n!(m-n)!}$
$P_\nu^\mu(x)$	Associated Legendre function of the first kind	
$(p)_n$	Pochhammer's symbol	$\frac{\Gamma(p+n)}{\Gamma(p)}$
$si(x)$	Sine integral	$-\int_x^\infty \frac{\sin t}{t}\,dt$
sgn x	Sign of the real number	$x/\lvert x\rvert$
$W_{p,\,q}(x)$	Whittaker function	
$Y_p(x)$	Neumann function	
γ	Euler's constant	0.57721 56649 . . .
$\gamma(a, x)$	Incomplete gamma function	$\int_0^x e^{-t}t^{a-1}dt$
$\Gamma(a, x)$	Incomplete gamma function	$\int_x^\infty e^{-t}t^{a-1}dt$
$\Gamma(a)$	Gamma function	$\int_0^\infty e^{-t}t^{a-1}dt$
$\zeta(p)$	Riemann zeta function	$\sum_{m=0}^{\infty}\frac{1}{(m+1)^p}$

$\Phi(x,p,q)$		$\sum_{m=0}^{\infty}\frac{x^m}{(m+q)^p}$
$\psi(p)$	Psi function	$\frac{d}{dp}\ln\Gamma(p)$
$\psi(a;b;x)$	Confluent hypergeometric function.	

3. Definition, Special Values, and Integral Representations

3.1. Definition and Other Notations

1. $E_1(x)=\int_x^{\infty}\frac{e^{-t}}{t}dt \qquad x>0$

2. $Ei(x)=-\int_{-x}^{\infty}\frac{e^{-t}}{t}dt$

$$=\int_{-\infty}^{x}\frac{e^{t}}{t}dt \qquad x>0.$$

3. Some authors use $[-Ei(-x)]$ for $E_1(x)$.
4. Some authors use $E^*(x)$ or $\bar{E}i(x)$ for $Ei(x)$.
5. Integrals involving $li(x)$ can be transformed into integrals over $Ei(x)$ since

$$li(x)=\int_0^x\frac{dt}{\ln t}=Ei(\ln x) \qquad x>1.$$

3.2. Special Values

1. $Ei\ (0.372507\ .\ .\ .)=0.$

2. $Ei(x)=E_1(x)$ at $x=0.523823\ .\ .\ .\ .$

3. $\lim_{x\to 0}[x^pE_1(x)]=\lim_{x\to 0}(x^pEi(x))=0 \qquad p>0.$

4. $\lim_{x\to 0}[\ln x+E_1(x)]=-\gamma.$

5. $\lim_{x\to\infty}[x^pE_1(x)]=\lim_{x\to\infty}[\ln x E_1(x)]=0.$

6. $\lim_{x\to\infty}[e^{-x}Ei(x)]=0.$

7. Inflection point of $Ei(x)$ at $x=1$.

3.3. Integral Representations

1. $E_1(x)=-\gamma-\ln x+\int_0^x(1-e^{-t})\frac{dt}{t}$

2. $E_1(x)=-\gamma-e^{-x}\ln x-\int_0^x e^{-t}\ln t\ dt$

3. $E_1(x)=x\int_1^{\infty}e^{-xt}\ln t\ dt$

4. $E_1(x)=e^{-x}\int_0^1\frac{1}{(x-\ln t)}dt$

5. $E_1(x)=e^{-x}\int_1^{\infty}\frac{1}{(x+\ln t)}\frac{dt}{t^2}$

6. $E_1(x)=e^{-x}\int_0^\infty \frac{e^{-xt}}{(1+t)}\,dt$

7. $E_1(x)=\frac{e^{-x}}{x}\left[1-\int_0^\infty \frac{x}{(x+t)^2}e^{-t}dt\right]$

8. $E_1(x)=\frac{e^{-x}}{x}\left[1-\int_0^1 \frac{t^{x-1}}{(1-\ln t)^2}\,dt\right]$

9. $E_1(x)=e^{-x}\int_0^\infty \ln\left(1+\frac{t}{x}\right)e^{-t}dt$

10. $E_1(x)=\frac{1}{\pi}\int_0^\infty \sin t\,\ln\left(1+\frac{t^2}{x^2}\right)\frac{dt}{t}$

11. $E_1(x)=2e^{-x}\int_0^\infty K_0(2\sqrt{xt})e^{-t}dt$

12. $E_1(x)=e^{-x}\int_0^\infty \frac{1}{(t-ix)}e^{-it}dt$

13. $E_1(x)=\int_0^\infty \exp(-xe^t)\,dt$

14. $Ei(x)=e^x\int_0^1 \frac{1}{(x+\ln t)}dt$

15. $Ei(x)=e^x\int_1^\infty \frac{1}{(x-\ln t)}\frac{dt}{t^2}$

16. $Ei(x)=e^x\int_0^\infty \frac{e^{-xt}}{(1-t)}\,dt$

17. $Ei(x)=\frac{e^x}{x}\left[\int_0^\infty \frac{xe^{-t}}{(x-t)^2}\,dt+1\right]$

18. $Ei(x)=\frac{e^x}{x}\left[\int_0^1 \frac{t^{x-1}}{(1+\ln t)^2}\,dt+1\right]$

19. $Ei(x)=-e^x\int_0^\infty \frac{e^{-it}}{(t+ix)}\,dt$

20. $[E_1(x)]^2=2e^{-x}\int_1^\infty \frac{e^{-xt}}{(1+t)}\ln t\,dt.$

21. $E_1(ax)E_1(bx)+E_1[(a+b)x]\ln(ab)=e^{-(a+b)x}\int_0^\infty \frac{e^{-xt}}{(a+b+t)}\ln[(a+t)(b+t)]\,dt$

22. $e^{-x}Ei(x)+e^xE_1(x)=2\int_0^\infty \frac{x}{(t^2+x^2)}\sin t\,dt$

23. $e^{-x}Ei(x)+e^xE_1(x)=2e^{-x}\ln x-\frac{4}{\pi}\int_0^\infty \frac{x}{(t^2+x^2)}\cos t\,\ln t\,dt$

24. $e^{-x}Ei(x)-e^xE_1(x)=-2\int_0^\infty \frac{t}{(t^2+x^2)}\cos t\,dt$

25. $e^{-x}Ei(x)-e^xE_1(x)=2e^{-x}\ln x-\frac{4}{\pi}\int_0^\infty \frac{t}{(t^2+x^2)}\sin t\,\ln t\,dt$

4. Integrals of the Exponential Integral With Other Functions

4.1. Combination of Exponential Integral With Powers

1. $\int E_1(ax)dx = xE_1(ax) - \frac{1}{a}e^{-ax}$

2. $\int Ei(ax)dx = xEi(ax) - \frac{1}{a}e^{ax}$

3. $\int_0^\infty E_1(ax)dx = \frac{1}{a}$

4. $\int xE_1(ax)dx = \frac{1}{2}x^2E_1(ax) - \frac{1}{2a^2}(1+ax)e^{-ax}$

5. $\int_0^\infty xE_1(ax)dx = \frac{1}{2a^2}$

6. $\int x^nE_1(ax)dx = \frac{x^{n+1}}{(n+1)}E_1(ax) - \frac{n!}{(n+1)}\frac{1}{a^{n+1}}e_n(ax)e^{-ax}$

7. $\int x^nE_1(ax+b)dx = E_1(ax+b)\sum_{m=0}^{n}(-1)^m\frac{n!}{(n-m)!}\frac{x^{n-m}}{(m+1)!}\left(x+\frac{b}{a}\right)^{m+1}$

$$-e^{-(ax+b)}\sum_{m=0}^{n}\frac{n!}{(n-m)!}\frac{1}{(m+1)!}\frac{x^{n-m}}{a^{m+1}}\sum_{k=0}^{m}(-1)^k(m-k)!(ax+b)^k$$

8. $\int_0^\infty x^nE_1(ax)dx = \frac{n!}{(n+1)}\cdot\frac{1}{a^{n+1}}$

9. $\int_0^\infty x^nE_1(ax+b)dx = \frac{1}{(n+1)}\cdot\frac{1}{a^{n+1}}\left[(-1)^{n+1}b^{n+1}E_1(b)+\sum_{m=0}^{n}(-1)^mb^m(n-m)!e^{-b}\right]$

10. $\int_a^\infty x^nE_1(x-y)dx = e^{-(a-y)}\sum_{m=0}^{n}\binom{n}{m}\frac{m!}{(m+1)}y^{n-m}e_m(a-y)$

$$-E_1(a-y)\sum_{m=0}^{n}\binom{n}{m}\frac{y^{n-m}}{(m+1)}(a-y)^{m+1} \qquad y<a$$

11. $\int_a^\infty x^nE_1(x-a)dx = \sum_{m=0}^{n}\binom{n}{m}\frac{m!}{(m+1)}a^{n-m}$

12. $\int_a^\infty x^nE_1(|y-x|)dx = \sum_{m=0}^{n}\binom{n}{m}\frac{m!}{(m+1)}[1+(-1)^m]y^{n-m}$

$$-e^{-(y-a)}\sum_{m=0}^{n}\binom{n}{m}\frac{m!}{(m+1)}(-1)^my^{n-m}e_m(y-a)$$

$$+E_1(y-a)\sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^m}{(m+1)}y^{n-m}(y-a)^{m+1} \qquad y>a$$

13. $\int_0^\infty x^nE_1(|y-x|)dx = \sum_{m=0}^{n}\binom{n}{m}\frac{m!}{(m+1)}[1+(-1)^m]y^{n-m}$

$$-e^{-y}\sum_{m=0}^{n}\binom{n}{m}\frac{m!}{(m+1)}(-1)^my^{n-m}e_m(y)+\frac{y^{n+1}}{(n+1)}E_1(y)$$

14. $\int x^p E_1(ax)dx = \frac{x^{p+1}}{(p+1)} E_1(ax) + \frac{1}{(p+1)} \frac{1}{a^{p+1}} \gamma(p+1, ax) \qquad p > -1$

15. $\int_0^\infty x^p E_1(ax)dx = \frac{1}{(p+1)} \frac{1}{a^{p+1}} \Gamma(p+1) \qquad p > -1$

16. $\int_0^\infty x^p E_1(x+b)dx = \Gamma(p+1) e^{-b/2} b^{p/2} W_{-(p+2)/2,\, -(p+1)/2}(b) \qquad p > -1$

17. $\int E_1(ax) \frac{dx}{x} = \int e^{-ax} \ln x \frac{dx}{x} + \ln x E_1(ax)$

18. $\int_b^\infty E_1(ax) \frac{dx}{x} = \frac{1}{2} [(\gamma + \ln ab)^2 + \zeta(2)] + \sum_{n=1}^{\infty} \frac{(-ab)^n}{n! n^2}$

19. $\int_b^\infty E_1(x+b) \frac{dx}{x} = \frac{1}{2} [E_1(b)]^2$

20. $\int_b^\infty E_1(ax) \frac{dx}{x^2} = \frac{1}{b} [(1+ab) E_1(ab) - e^{-ab}]$

21. $\int E_1(ax+b) \frac{dx}{x^2} = \frac{1}{b} [a e^{-b} E_1(ax) - \frac{1}{x} (ax+b) E_1(ax+b)]$

22. $\int E_1(ax) \frac{dx}{x^{n+2}} = -\frac{1}{(n+1)} \left[\frac{1}{x^{n+1}} E_1(ax) + \int e^{-ax} \frac{dx}{x^{n+2}} \right]$

23. $\int_b^\infty E_1(ax) \frac{dx}{x^{n+2}} = \frac{1}{(n+1)(n+1)! b^{n+1}} [\{(n+1)! + (-1)^n (ab)^{n+1}\} E_1(ab)$

$$- e^{-ab} \sum_{m=0}^{n} (n-m)! (-ab)^m]$$

24. $\int E_1(ax+b) \frac{dx}{x^{n+2}} = -\frac{1}{(n+1)} \left\{ \frac{1}{x^{n+1}} + (-1)^n \left(\frac{a}{b} \right)^{n+1} \right\} E_1(ax+b)$

$$+ \frac{(-1)^n}{(n+1)} \left(\frac{a}{b} \right)^{n+1} e_n(b) e^{-b} E_1(ax)$$

$$+ \frac{(-1)^n}{(n+1)} \left(\frac{a}{b} \right)^{n+1} e^{-(ax+b)} \sum_{m=1}^{n} \frac{1}{m!} \left(-\frac{b}{ax} \right)^m \sum_{k=0}^{m-1} (m-k-1)! (-ax)^k$$

25. $\int E_1(ax) \frac{dx}{x^p} = -\frac{1}{(p-1)} \left[\frac{1}{x^{p-1}} E_1(ax) + \int e^{-ax} \frac{dx}{x^p} \right] \qquad p > 1$

26. $\int_b^\infty E_1(ax) \frac{dx}{x^{n+p+1}} = \frac{1}{(n+p)} \cdot \frac{1}{b^{n+p}} E_1(ab) + (-1)^n \frac{a^{n+p}}{(n+p)} \frac{\Gamma(p)}{\Gamma(n+p+1)} \Gamma(1-p, ab)$

$$- \frac{1}{(n+p)} \cdot \frac{1}{b^{n+p}} \cdot \frac{e^{-ab}}{\Gamma(n+p+1)} \sum_{m=0}^{n} (-ab)^m \Gamma(n-m+p) \qquad 0 < p < 1$$

27. $\int_0^\infty E_1(ax^2)\, dx = \sqrt{\frac{\pi}{a}}$

28. $\int_0^\infty x^p (x+a)^{-p-2} E_1(bx) dx = \frac{1}{a} \cdot \frac{1}{\sqrt{ab}} \cdot \frac{\Gamma(p+1)}{(p+1)} e^{ab/2} W_{-(p+1/2),0}(ab) \qquad p > -1$

29. $\int_0^\infty (x+a)^{-2} E_1(bx) dx = \frac{1}{a} e^{ab} E_1(ab)$

30. $\int_0^\infty x^p(x+a)^{-p}\left\{1+\frac{2x}{a(p+1)}\right\}E_1(x)\,dx=\frac{1}{\sqrt{a}}\cdot\frac{\Gamma(p+1)}{(p+1)}\,e^{a/2}W_{1/2-p,\,-1}(a) \qquad p>-1$

31. $\int_0^\infty x^p(x+a)^p\left\{1+\frac{2x}{a}\right\}E_1(x)\,dx=a^{p-1/2}\,\frac{\Gamma(p+1)}{(p+1)}\,e^{a/2}W_{1/2,\,-p-1}(a) \qquad p>-1$

32. $\int_0^\infty x^p(x+a)^{-p-2}E_1(x+a)dx=a^{-3/2}\Gamma(p+1)e^{-a/2}W_{-(p+3/2),\,0}(a) \qquad p>-1$

4.2. Combination of Exponential Integral With Exponentials and Powers

1. $\int e^{-ax}E_1(bx)dx=\frac{1}{a}\,[E_1\{(a+b)x\}-e^{-ax}E_1(bx)]$

2. $\int e^{ax}E_1(bx)dx=-\frac{1}{a}\,[E_1\{(b-a)x\}-e^{ax}E_1(bx)] \qquad b>a$

3. $\int_0^\infty e^{-ax}E_1(bx)dx=\frac{1}{a}\ln\left(1+\frac{a}{b}\right)$

4. $\int_0^\infty e^{ax}E_1(bx)dx=-\frac{1}{a}\ln\left(1-\frac{a}{b}\right) \qquad b>a$

5. $\int_0^c e^{ax}E_1(ax)dx=\frac{1}{a}\,[\gamma+\ln\,(ac)+e^{ac}E_1(ac)]$

6. $\int_0^c e^{ax}E_1(bx)dx=-\frac{1}{a}\left[E_1\{(a-b)c\}-e^{ac}E_1(bc)+\ln\left(\frac{a}{b}-1\right)\right] \qquad a>b$

7. $\int e^{-ax}Ei(bx)dx=-\frac{1}{a}\,[E_1\{(a-b)x\}+e^{-ax}Ei(bx)] \qquad a>b$

8. $\int_0^\infty e^{-ax}Ei(bx)dx=-\frac{1}{a}\ln\left(\frac{a}{b}-1\right) \qquad a>b$

9. $\int_0^c e^{-ax}Ei(ax)dx=\frac{1}{a}\,[\gamma+\ln\,(ac)-e^{-ac}Ei(ac)]$

10. $\int xe^{-ax}E_1(bx)dx=\frac{1}{a^2}\left[E_1\{(a+b)x\}-(1+ax)e^{-ax}E_1(bx)+\left(\frac{a}{a+b}\right)e^{-(a+b)x}\right]$

11. $\int_0^\infty xe^{-ax}E_1(bx)dx=\frac{1}{a^2}\left[\ln\left(1+\frac{a}{b}\right)-\frac{a}{a+b}\right]$

12. $\int_0^c xe^{ax}E_1(ax)dx=\frac{1}{a^2}\,[ac-\gamma-\ln\,(ac)-(1-ac)e^{ac}E_1(ac)]$

13. $\int xe^{cx}E_1(ax+b)dx=\frac{1}{c}\left(x-\frac{1}{c}\right)e^{cx}E_1(ax+b)-\frac{1}{c(a-c)}\,e^{-\{(a-c)x+b\}}$
$+\frac{1}{ac^2}\,(a+bc)e^{-bc/a}E_1\left\{\frac{(a-c)(ax+b)}{a}\right\} \qquad a>c$

14. $\int xe^{ax}E_1(ax+b)dx=\frac{1}{a}\left(x-\frac{1}{a}\right)e^{ax}E_1(ax+b)+\frac{1}{a}\left\{x-\frac{1}{a}\,(1+b)\,\ln\,(ax+b)\right\}e^{-b}$

15. $\int_0^\infty xe^{-ax}Ei(bx)dx=-\frac{1}{a^2}\left[\ln\left(\frac{a}{b}-1\right)-\frac{a}{a-b}\right] \qquad a>b$

16. $$\int x^n e^{-ax} E_1(bx)\,dx = \frac{n!}{a^{n+1}} E_1\{(a+b)x\} - \frac{n!}{a^{n+1}} e_n(ax) e^{-ax} E_1(bx) + \frac{n!}{a^{n+1}} e^{-(a+b)x} \sum_{m=1}^{n} \frac{e_{m-1}\{(a+b)x\}}{m\left(1+\frac{b}{a}\right)^m}$$

17. $$\int_0^\infty x^n e^{-ax} E_1(bx)\,dx = \frac{n!}{a^{n+1}}\left[\ln\left(1+\frac{a}{b}\right) - \sum_{m=1}^{n} \frac{1}{m}\left(\frac{a}{a+b}\right)^m\right]$$

18. $$\int_0^\infty x^n e^{-ax} Ei(bx)\,dx = \frac{-n!}{a^{n+1}}\left[\ln\left(\frac{a}{b}-1\right) - \sum_{m=1}^{n} \frac{1}{m}\left(\frac{a}{a-b}\right)^m\right] \qquad a > b$$

19. $$\int x^p e^{-ax} E_1(bx)\,dx = \frac{1}{a^{p+1}}\gamma(p+1, ax) E_1(bx) + \frac{1}{b^{p+1}} \sum_{m=0}^{\infty} \frac{\gamma(p+m+1, bx)}{m!(p+m+1)}\left(\frac{-a}{b}\right)^m \qquad p > -1$$

20. $$\int_0^\infty x^p e^{-ax} E_1(bx)\,dx = \frac{\Gamma(p+1)}{p+1} \cdot \frac{1}{(a+b)^{p+1}} {}_2F_1\left(1, p+1; p+2; \frac{a}{a+b}\right)$$
$$= \frac{\Gamma(p+1)}{a^{p+1}} B_{a/(a+b)}(p+1, 0)$$
$$= \frac{\Gamma(p+1)}{(a+b)^{p+1}} \sum_{m=0}^{\infty} \frac{1}{p+m+1}\left(\frac{a}{a+b}\right)^m \qquad p > -1$$

21. $$\int_0^\infty x^p e^{ax} E_1(bx)\,dx = \frac{\Gamma(p+1)}{(p+1)} \cdot \frac{1}{b^{p+1}} {}_2F_1\left(p+1, p+1; p+2; \frac{a}{b}\right) \qquad b > a,\ p > -1$$

22. $$\int_0^\infty x^p e^{ax} E_1(ax)\,dx = -\frac{\pi}{\sin(p\pi)} \frac{\Gamma(p+1)}{a^{p+1}} \qquad -1 < p < 0$$

23. $$\int_0^\infty x^p e^{-ax} Ei(ax)\,dx = -\pi \cot(p\pi) \frac{\Gamma(p+1)}{a^{p+1}} \qquad -1 < p < 0$$

24. $$\int_0^\infty x^{p-1} e^x E_1(x+a)\,dx = \Gamma(p)\Gamma(1-p) e^{-a/2} a^{(p-1)/2} W_{(p-1)/2,\, p/2}(a) \qquad 0 < p < 1$$

25. $$\int_{-\ln b}^\infty e^{-ax}\{E_1(e^{-x}) - E_1(b)\}\,dx = \frac{1}{a}\gamma(a, b) \qquad b < 1$$

26. $$\int_0^\infty x^{-1/2} e^{-ax} E_1(bx)\,dx = 2\sqrt{\frac{\pi}{a}} \ln\left(\sqrt{\frac{a}{b}} + \sqrt{\frac{a+b}{b}}\right)$$

27. $$\int_0^\infty x^{-1/2} e^{ax} E_1(bx)\,dx = 2\sqrt{\frac{\pi}{a}} \sin^{-1}\left(\sqrt{\frac{a}{b}}\right) \qquad b \geq a$$

28. $$\int e^{-ax} E_1(bx)\frac{dx}{x} = -\int e^{-bx} E_1(ax)\frac{dx}{x} - E_1(ax) E_1(bx)$$

29. $$\int_c^\infty e^{-ax} E_1(bx)\frac{dx}{x} = [\gamma + \ln ac + E_1(ac)] E_1(bc) + \frac{1}{2}[\zeta(2) + (\gamma + \ln bc)^2] + e^{-bc} \sum_{m=0}^{\infty} \frac{e_m(bc)}{(m+1)^2}\left(-\frac{a}{b}\right)^{m+1} + \sum_{m=1}^{\infty} \frac{(-bc)^m}{m!m^2}$$

30. $$\int e^{-ax} E_1(ax)\frac{dx}{x} = -\frac{1}{2}[E_1(ax)]^2$$

31. $\int_0^\infty (1-e^{-ax})E_1(bx)\,\frac{dx}{x}=\int_0^{a/b}\ln(1+x)\,\frac{dx}{x}$

$$=-L_2\left(-\frac{a}{b}\right)$$

$$=-\sum_{m=0}^{\infty}\frac{(-a/b)^{m+1}}{(m+1)^2} \qquad b\geq a$$

32. $\int_0^\infty\left[\frac{e^{ax}E_1(ax)}{x+b}-\frac{e^{-ax}Ei(ax)}{x-b}\right]dx=\pi^2e^{-ab} \qquad a>0$

$$=0 \qquad a<0$$

33. $\int_0^a (a-x)^{p-1}x^{-p}e^xE_1(x)dx=a^{-1/2}\Gamma(p)\Gamma(1-p)\Gamma(1-p)W_{p-1/2,\,0}(a) \qquad 0<p<1$

34. $\int_{-\infty}^\infty e^{ax}e^{-ibx}E_1(ax)dx=\frac{\pi}{(b+ia)}\operatorname{sgn}b$

35. $\int_{-\infty}^\infty e^{-ax}e^{-ibx}Ei(ax)dx=\frac{-\pi}{(b-ia)}\operatorname{sgn}b$

36. $\int_0^\infty e^{-ax^2}E_1(bx^2)dx=\sqrt{\frac{\pi}{a}}\ln\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{a+b}{b}}\right)$

37. $\int_0^\infty e^{ax^2}E_1(bx^2)dx=\sqrt{\frac{\pi}{a}}\sin^{-1}\left(\sqrt{\frac{a}{b}}\right) \qquad b\geq a$

38. $\int_0^\infty e^{-ax}E_1\left(\frac{b}{x}\right)dx=\frac{2}{a}K_0(2\sqrt{ab})$

39. $\int_0^\infty e^{-ax^2}E_1\left(\frac{b}{x^2}\right)dx=\sqrt{\frac{\pi}{a}}E_1(2\sqrt{ab})$

40. $\int_0^\infty e^{-ax^2}E_1\left(\frac{b}{x^2}\right)\frac{dx}{x^2}=\sqrt{\frac{\pi}{b}}e^{-2\sqrt{ab}}-2\sqrt{a\pi}E_1(2\sqrt{ab})$

41. $\int_0^\infty e^{-a^2x^2}e^{b^2/x^2}E_1\left(\frac{b^2}{x^2}\right)dx=-\frac{\sqrt{\pi}}{a}[\cos(2ab)Ci(2ab)+\sin(2ab)si(2ab)]$

42. $\int_0^\infty e^{-a^2x^2}e^{b^2/x^2}E_1\left(\frac{b^2}{x^2}\right)\frac{dx}{x^2}=-\frac{\sqrt{\pi}}{b}[\cos(2ab)si(2ab)-\sin(2ab)Ci(2ab)]$

43. $\int_0^\infty\cosh(ax)E_1(bx)dx=\frac{1}{2a}\ln\left(\frac{b+a}{b-a}\right) \qquad b>a$

44. $\int_0^\infty\sinh(ax)E_1(bx)\,\frac{dx}{x}=\sum_{m=0}^{\infty}\frac{(a/b)^{2m+1}}{(2m+1)^2} \qquad b\geq a$

4.3. Combination of Exponential Integral With Trigonometric Functions

1. $\int_0^\infty E_1(ax)\sin(bx)dx=\frac{1}{2b}\ln\left(1+\frac{b^2}{a^2}\right)$

2. $\int_0^\infty E_1(ax)\cos(bx)dx=\frac{1}{b}\tan^{-1}\left(\frac{b}{a}\right)$

3. $\int_0^\infty E_1(ax)\,\frac{\sin(2nx)}{\sin x}\,dx = 2\sum_{m=0}^{n-1}\frac{1}{(2m+1)}\tan^{-1}\left(\frac{2m+1}{a}\right)$

4. $\int_0^\infty E_1(ax)\,\frac{\sin[(2n+1)x]}{\sin x}\,dx = \frac{1}{a} + \sum_{m=1}^{n}\frac{1}{m}\tan^{-1}\left(\frac{2m}{a}\right)$

5. $\int_0^\infty \sin(a\sqrt{x})\,E_1(x)dx = \frac{2\pi}{a^2}\,\mathrm{erf}\left(\frac{a}{2}\right) - \frac{2}{a}\sqrt{\pi}\,e^{-a^2/4}$

6. $\int_0^\infty x^{-1/2}\cos(a\sqrt{x})\,E_1(x)dx = \frac{2\pi}{a}\,\mathrm{erf}\left(\frac{a}{2}\right)$

7. $\int_0^\infty x^{p-1/2}\sin(a\sqrt{x})\,E_1(x)dx = a\,\frac{\Gamma(p+1)}{(p+1)}\,{}_2F_2\left(p+1, p+1; \frac{3}{2}, p+2; -\frac{a^2}{4}\right) \qquad p > -1$

8. $\int_0^\infty x^p\cos(a\sqrt{x})\,E_1(x)dx = \frac{\Gamma(p+1)}{(p+1)}\,{}_2F_2\left(p+1, p+1; \frac{1}{2}, p+2; -a^2/4\right) \qquad p > -1$

9. $\int_0^\infty E_1(ax)\sin bx e^{-cx}dx = \frac{1}{(b^2+c^2)}\left[\frac{b}{2}\ln\left\{\frac{(a+c)^2+b^2}{a^2}\right\} - c\tan^{-1}\left(\frac{b}{a+c}\right)\right]$

10. $\int_0^\infty E_1(ax)\sin(bx)e^{cx}dx = \frac{1}{(b^2+c^2)}\left[\frac{b}{2}\ln\left\{\frac{(a-c)^2+b^2}{a^2}\right\} + c\tan^{-1}\left(\frac{b}{a-c}\right)\right] \qquad a \geqslant c$

11. $\int_0^\infty Ei(ax)\sin(bx)e^{-cx}dx = \frac{1}{(b^2+c^2)}\left[-\frac{b}{2}\ln\left\{\frac{(c-a)^2+b^2}{a^2}\right\} + c\tan^{-1}\left(\frac{b}{c-a}\right)\right] \qquad c \geqslant a$

12. $\int_0^\infty E_1(ax)\cos(bx)e^{-cx}dx = \frac{1}{(b^2+c^2)}\left[\frac{c}{2}\ln\left\{\frac{(a+c)^2+b^2}{a^2}\right\} + b\tan^{-1}\left(\frac{b}{a+c}\right)\right]$

13. $\int_0^\infty E_1(ax)\cos(bx)e^{cx}dx = \frac{1}{(b^2+c^2)}\left[-\frac{c}{2}\ln\left\{\frac{(a-c)^2+b^2}{a^2}\right\} + b\tan^{-1}\left(\frac{b}{c-a}\right)\right] \qquad a \geqslant c$

14. $\int_0^\infty Ei(ax)\cos(bx)e^{-cx}dx = \frac{1}{(b^2+c^2)}\left[-\frac{c}{2}\ln\left\{\frac{(c-a)^2+b^2}{a^2}\right\} - b\tan^{-1}\left(\frac{b}{c-a}\right)\right] \qquad c \geqslant a$

15. $\int_0^\infty E_1(ax)\sin^2\left(\frac{1}{2}bx\right)e^{-cx}dx = \frac{1}{2c}\ln\left(1+\frac{c}{a}\right) - \frac{1}{2}\cdot\frac{1}{(b^2+c^2)} \times\left[\frac{c}{2}\ln\left\{\frac{(a+c)^2+b^2}{a^2}\right\} + b\tan^{-1}\left(\frac{b}{a+c}\right)\right]$

16. $\int_0^\infty E_1(ax)\cos^2\left(\frac{1}{2}bx\right)e^{-cx}dx = \frac{1}{2c}\ln\left(1+\frac{c}{a}\right) + \frac{1}{2}\cdot\frac{1}{(b^2+c^2)} \times\left[\frac{c}{2}\ln\left\{\frac{(a+c)^2+b^2}{a^2}\right\} + b\tan^{-1}\left(\frac{b}{a+c}\right)\right].$

4.4. Combination of Exponential Integral With Logarithms and Powers

1. $\int \ln x E_1(bx)dx = \frac{1}{b}[(1-\ln x)e^{-bx} - (1+bx-bx\ln x)E_1(bx)]$

2. $\int_0^\infty \ln x E_1(bx)dx = -\frac{1}{b}(1+\gamma+\ln b)$

3. $\int \ln x Ei(bx)dx = \frac{1}{b}[(1-\ln x)e^{bx} + (1-bx+bx\ln x)Ei(bx)]$

4. $$\int x \ln x E_1(bx)\,dx = \frac{1}{2b^2}\left\{\frac{1}{2}(1+bx) - (1+bx)\ln x - 1\right\} e^{-bx} - \frac{1}{2b^2}\left(1 + \frac{1}{2}b^2x^2 - b^2x^2 \ln x\right) E_1(bx)$$

5. $$\int_0^\infty x \ln x E_1(bx)\,dx = -\frac{1}{2b^2}\left(-\frac{1}{2} + \gamma + \ln b\right)$$

6. $$\int x^n \ln x E_1(bx)\,dx = \frac{n!}{(n+1)b^{n+1}}\left[e_n(bx)\left(\frac{1}{n+1} - \ln x\right) - \sum_{m=0}^{n-1}\frac{e_m(bx)}{(m+1)}\right] e^{-bx} - \frac{n!}{(n+1)b^{n+1}}\left[1 + \frac{(bx)^{n+1}}{(n+1)!}\{1 - (n+1)\ln x\}\right] E_1(bx)$$

7. $$\int_0^\infty x^n \ln x E_1(bx)\,dx = \frac{-n!}{(n+1)b^{n+1}}\left[\gamma + \ln b + \frac{1}{n+1} - \sum_{m=1}^{n}\frac{1}{m}\right]$$

8. $$\int x^p \ln x E_1(bx)\,dx = \frac{1}{(p+1)b^{p+1}}\left\{\ln x - \frac{1}{(p+1)}\right\}[\gamma(p+1, bx) + (bx)^{p+1}E_1(bx)] - \frac{x^{p+1}}{p+1}\sum_{m=0}^{\infty}\frac{(-bx)^m}{m!(p+m+1)^2} \qquad p > -1$$

9. $$\int_0^\infty x^p \ln x E_1(bx)\,dx = \frac{-\Gamma(p+1)}{(p+1)} \cdot \frac{1}{b^{p+1}}\left[\ln b + \frac{1}{p+1} - \Psi(p+1)\right] \qquad p > -1.$$

4.5. Combination of Exponential Integral With Logarithms, Exponentials, and Powers

1. $$\int e^{-ax} \ln x E_1(bx)\,dx = -\frac{1}{a}\int e^{-bx}E_1(ax)\frac{dx}{x} + \frac{\ln x}{a}\left[\gamma + \ln\{(a+b)x\} + E_1\{(a+b)x\}\right] - \frac{1}{a}\left[\ln x\, e^{-ax} + E_1(ax)\right]E_1(bx) - \frac{1}{2a}\ln^2 x + \frac{1}{a}\sum_{m=1}^{\infty}\frac{\{-(a+b)x\}^m}{m!m^2}$$

2. $$\int_0^\infty e^{-ax} \ln x E_1(bx)\,dx = -\frac{1}{a}\left[\ln\left(1+\frac{a}{b}\right)\{\gamma + \ln(a+b)\} + \left(\frac{a}{a+b}\right)\Phi\left(\frac{a}{a+b}, 2, 1\right)\right]$$

3. $$\int_0^\infty e^{-ax} \ln x E_1(ax)\,dx = -\frac{1}{2a}\left[\zeta(2) + (\gamma + \ln a)\ln 4 + \ln^2 2\right]$$

4. $$\int_0^\infty x e^{-ax} \ln x E_1(bx)\,dx = -\frac{1}{a^2}\left[\left\{\ln\left(1+\frac{a}{b}\right) - \frac{a}{a+b}\right\}(\gamma + \ln(a+b) - 1) + \left(\frac{a}{a+b}\right)^2\Phi\left(\frac{a}{a+b}, 2, 2\right)\right]$$

5. $$\int_0^\infty x^n e^{-ax} \ln x E_1(bx)\,dx = -\frac{n!}{a^{n+1}}\left\{\ln\left(1+\frac{a}{b}\right) - \sum_{m=1}^{n}\frac{1}{m}\left(\frac{a}{a+b}\right)^m\right\}\left[\gamma + \ln(a+b) - \sum_{m=1}^{n}\frac{1}{m}\right] - \frac{n!}{(a+b)^{n+1}}\Phi\left(\frac{a}{a+b}, 2, n+1\right)$$

6. $$\int_0^\infty x^p e^{-ax} \ln x E_1(bx)\,dx = \frac{\Gamma(p+1)}{(a+b)^{p+1}}\left[\{\Psi(p+1) - \ln(a+b)\}\Phi\left(\frac{a}{a+b}, 1, p+1\right) - \Phi\left(\frac{a}{a+b}, 2, p+1\right)\right]$$

$p > -1.$

1. $$\int E_1(ax)E_1(bx)dx = xE_1(ax)E_1(bx) + \left(\frac{1}{a}+\frac{1}{b}\right)E_1\{(a+b)x\} - \frac{1}{a}e^{-ax}E_1(bx) - \frac{1}{b}e^{-bx}E_1(ax)$$

2. $$\int_0^\infty E_1(ax)E_1(bx)dx = \left(\frac{1}{a}+\frac{1}{b}\right)\ln(a+b) - \frac{1}{a}\ln b - \frac{1}{b}\ln a$$

3. $$\int E_1(ax)E_1(bx)dx = xE_1(ax)E_i(bx) + \left(\frac{1}{b}-\frac{1}{a}\right)E_1\{(a-b)x\} - \frac{1}{a}e^{-ax}Ei(bx) - \frac{1}{b}e^{bx}E_1(ax) \qquad a>b$$

4. $$\int_0^\infty E_1(ax)\,Ei(bx)dx = \frac{1}{a}\left[\left(\frac{a-b}{b}\right)\ln\left(\frac{a-b}{b}\right) - \left(\frac{a}{b}\right)\ln\left(\frac{a}{b}\right)\right] \qquad a>b$$

5. $$\int E_1(ax)Ei(ax)dx = xE_1(ax)Ei(ax) - \frac{1}{a}[e^{-ax}Ei(ax) + e^{ax}E_1(ax)]$$

6. $$\int_0^\infty E_1(ax)Ei(ax)dx = 0$$

7. $$\int xE_1(ax)E_1(bx)dx = \frac{x^2}{2}E_1(ax)E_1(bx) + \frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)E_1\{(a+b)x\} - \frac{1}{2a^2}e_1(ax)e^{-ax}E_1(bx) - \frac{1}{2b^2}e_1(bx)e^{-bx}E_1(ax) + \frac{1}{2ab}e^{-(a+b)x}$$

8. $$\int_0^\infty xE_1(ax)E_1(bx)dx = \frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\ln(a+b) - \frac{1}{2a^2}\ln b - \frac{1}{2b^2}\ln a - \frac{1}{2ab}$$

9. $$\int x^nE_1(ax)E_1(bx)dx = \frac{x^{n+1}}{(n+1)}E_1(ax)E_1(bx) + \frac{n!}{(n+1)}\left(\frac{1}{a^{n+1}}+\frac{1}{b^{n+1}}\right)E_1\{(a+b)x\} - \frac{n!}{a^{n+1}}\frac{1}{(n+1)}e_n(ax)e^{-ax}E_1(bx) - \frac{n!}{b^{n+1}}\frac{1}{(n+1)}e_n(bx)e^{-bx}E_1(ax) + \frac{n!}{(n+1)}e^{-(a+b)x}\sum_{m=1}^{n}\frac{e_{m-1}\{(a+b)x\}}{m(a+b)^m}\left[\frac{a^m}{a^{n+1}}+\frac{b^m}{b^{n+1}}\right]$$

10. $$\int_0^\infty x^nE_1(ax)\,E_1(bx)\,dx = -\frac{n!}{(n+1)}\left[\frac{1}{a^{n+1}}\left\{\ln\left(\frac{b}{a+b}\right) + \sum_{m=1}^{n}\frac{1}{m}\left(\frac{a}{a+b}\right)^m\right\} + \frac{1}{b^{n+1}}\left\{\ln\left(\frac{a}{a+b}\right) + \sum_{m=1}^{n}\frac{1}{m}\left(\frac{b}{a+b}\right)^m\right\}\right]$$

11. $$\int_0^\infty x^p[E_1(x)]^2dx = 2^{-p}\frac{\Gamma(p+1)}{(p+1)}\sum_{m=0}^{\infty}\frac{(1/2)^m}{(m+p+1)} = 2^{-p}\frac{\Gamma(p+1)}{(p+1)}\Phi\left(\frac{1}{2},1,p+1\right) \qquad p>-1$$

12. $$\int_0^a E_1(x)E_1(a-x)\,dx = 2(\gamma+\ln a)e^{-a}+2(1-a\gamma-a\ln a)E_1(a) - a\{\zeta(2)+(\gamma+\ln a)^2\}-2a\sum_{m=1}^{\infty}\frac{(-a)^m}{m!m^2}$$

13. $$\int_a^\infty E_1(x)E_1(x-a)\,dx = e^{-a}\{\ln 2 - e^{2a}E_1(2a)\}+\frac{1}{2}a\{[\gamma+\ln a]^2-2\zeta(2)\} - a(\gamma+\ln a)Ei(a)-a\ln 2\{E_1(a)+Ei(a)\} + E_1(a)+a\sum_{m=1}^{\infty} b_m\left(\frac{a^m}{m\cdot m!}\right)$$

where $$b_{2m}=\frac{1}{2m}+2\sum_{n=1}^{m}\frac{1}{(2n-1)},$$

and $$b_{2m+1}=\frac{1}{(2m+1)}+2\sum_{n=1}^{m+1}\frac{1}{(2n-1)}$$

14. $$\int_0^\infty x^{-1/2}E_1\left(\frac{x}{a^2}\right)E_1\left(\frac{a^2b^2}{4x}\right)dx = 4a\sqrt{\pi}\,[(1+b)E_1(b)-e^{-b}]$$

15. $$\int_0^\infty E_1\left(\frac{x^2}{a^2}\right)E_1\left(\frac{a^2b^2}{4x^2}\right)\,dx = 2a\sqrt{\pi}\,[(1+b)E_1(b)-e^{-b}]$$

16. $$c\int_0^\infty E_1(ax)E_1(bx)e^{-cx}\,dx = \frac{\pi^2}{6}-L_2\left(\frac{a}{a+b+c}\right)-L_2\left(\frac{b}{a+b+c}\right)+\ln a\ln b + \ln(a+c)\ln\left(\frac{a+b+c}{b}\right)+\ln(b+c)\ln\left(\frac{a+b+c}{a}\right) - \ln^2(a+b+c)$$

17. $$\int_0^\infty e^{-x}E_1(x)E_1(x)\,dx = \frac{\pi^2}{6}-2L_2\left(\frac{1}{3}\right)+2\ln 2\ln 3-\ln^2 3 = 1.228558\ldots$$

18. $$\int_0^\infty e^{x}E_1(x)E_1(x)\,dx = \frac{\pi^2}{6}$$

19. $$\int_0^\infty e^{2x}E_1(x)E_1(x)\,dx = \frac{\pi^2}{12}$$

20. $$a\int_0^\infty e^{ax}E_1(x)E_1\{(a+1)x\}\,dx = \frac{\pi^2}{12}+\frac{1}{2}\ln^2 2-L_2\left(\frac{1-a}{2}\right)+\ln 2\ln\left(\frac{1+a}{2}\right) = \frac{\pi^2}{12}+\frac{1}{2}\ln^2(1+a) + \sum_{m=1}^{\infty}\frac{(-1)^m}{m^2}\left(\frac{1-a}{1+a}\right)^m$$

21. $$a\int_0^\infty e^{2ax}E_1\{(a+1)x\}E_1\{(a+1)x\}\,dx = \frac{\pi^2}{12}+\sum_{m=1}^{\infty}\frac{(-1)^m}{m^2}\left(\frac{1-a}{1+a}\right)^m$$

22. $$\int_0^\infty e^{-x}E_1(x)Ei(x)\,dx = -\frac{\pi^2}{12}$$

23. $$a\int_0^\infty e^{-ax}E_1(x)\,Ei\{(a-1)x\}\,dx=-\frac{\pi^2}{12}-\frac{1}{2}\ln^2(a-1)+\ln(a+1)\,\ln(a-1)$$
$$-\sum_{m=1}^{\infty}\frac{(-1)^m}{m^2}\left(\frac{1-a}{1+a}\right)^m$$

24. $$\int_0^\infty e^{-2x}Ei(x)\,Ei(x)\,dx=\frac{\pi^2}{4}$$

25. $$c^2\int_0^\infty xE_1(ax)\,E_1(bx)\,e^{-cx}dx=\frac{\pi^2}{6}-L_2\left(\frac{a}{a+b+c}\right)-L_2\left(\frac{b}{a+b+c}\right)+\ln a\,\ln b$$
$$-\ln^2(a+b+c)+\left[\ln(a+c)-\frac{c}{a+c}\right]\ln\left(\frac{a+b+c}{b}\right)$$
$$+\left[\ln(b+c)-\frac{c}{b+c}\right]\ln\left(\frac{a+b+c}{a}\right)$$

26. $$\int_0^\infty xe^{-x}E_1(x)\,E_1(x)\,dx=\frac{\pi^2}{6}-2L_2\left(\frac{1}{3}\right)+2\ln 2\,\ln 3-\ln^2 3-\ln 3$$
$$=0.129946\ldots$$

27. $$\frac{c^{n+1}}{n!}\int_0^\infty x^nE_1(ax)\,E_1(bx)\,e^{-cx}dx=\frac{\pi^2}{6}-L_2\left(\frac{a}{a+b+c}\right)-L_2\left(\frac{b}{a+b+c}\right)$$
$$+\ln a\,\ln b-\ln^2(a+b+c)$$
$$+\left\{\ln(a+c)-\sum_{m=1}^{n}\frac{1}{m}\left(\frac{c}{a+c}\right)^m\right\}\ln\left(\frac{a+b+c}{b}\right)$$
$$+\left\{\ln(b+c)-\sum_{m=1}^{n}\frac{1}{m}\left(\frac{c}{b+c}\right)^m\right\}\ln\left(\frac{a+b+c}{a}\right)$$
$$+\sum_{m=2}^{n}\frac{1}{m}\left[\sum_{k=1}^{m-1}\frac{1}{(m-k)}\left\{\left(\frac{a+b+c}{a+c}\right)^k\right.\right.$$
$$\left.\left.+\left(\frac{a+b+c}{b+c}\right)^k\right\}\right]\left(\frac{c}{a+b+c}\right)^m$$

28. $$\int_0^\infty x^{-1/2}e^{x/a^2}e^{a^2b^2/(4x)}E_1\left(\frac{x}{a^2}\right)E_1\left(\frac{a^2b^2}{4x}\right)dx=2a\pi^{3/2}e^bE_1(b)$$

29. $$\int_0^\infty e^{x^2/a^2}e^{a^2b^2/(4x^2)}E_1\left(\frac{x^2}{a^2}\right)E_1\left(\frac{a^2b^2}{4x^2}\right)dx=a\pi^{3/2}e^bE_1(b)$$

30. $$\int_0^\infty E_1(x)\,dx\int_0^\infty E_1(y)\,e^{-a|x-y|}dy=\frac{4}{a}\ln 2-\frac{\pi^2}{6a^2}-\frac{1}{a^2}\ln^2(a+1)$$
$$+\frac{2}{a^2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m^2}\left(\frac{1-a}{1+a}\right)^m$$

31. $$\int_0^\infty \ln x\,[E_1(x)]^2dx=-[\zeta(2)+2(\gamma+1)\ln 2+\ln^2 2]$$

32. $$\int_0^\infty x\ln xE_1(x)E_1(x)dx=-\frac{1}{2}\left[\zeta(2)+\ln^2 2-\gamma-\frac{1}{2}+2(\gamma-1)\ln 2\right]$$

33. $$\int_0^\infty x^n\ln xE_1(x)E_1(x)dx=-\frac{n!}{(n+1)}\left[\zeta(2)+\ln^2 2-2\gamma\sum_{m=1}^{n}\frac{1}{2^m m}+2(\gamma+A)\ln 2-2B\right]$$

where

$$A=\frac{1}{n+1}-\sum_{m=1}^{n}\frac{1}{m}\left(1+\frac{1}{2^m}\right),$$

$$B=\sum_{m=1}^{n}\frac{1}{2^m m}\left[\frac{1}{n+1}+\frac{1}{m}-\sum_{k=1}^{n}\frac{1}{k}\right]$$

34. $$\int_0^\infty x^p \ln x E_1(x)E_1(x)dx=\frac{\Gamma(p+1)}{(p+1)}2^p\left[\left\{\Psi(p+1)-\frac{1}{p+1}-\ln 2\right\}\sum_{m=0}^{\infty}\frac{1}{2^m(m+p+1)}-\sum_{m=0}^{\infty}\frac{1}{2^m(m+p+1)^2}\right] \qquad p>-1$$

35. $$\int_0^\infty E_1(x)dx\int_0^\infty E_1(y)E_1(|x-y|)dy=4\ln 2-\frac{\pi^2}{6}$$

36. $$\int_0^\infty E_1(x)dx\int_0^x E_1(y)E_1(x-y)dy=\int_0^\infty E_1(x)dx\int_x^\infty E_1(y)E_1(y-x)dy=2\ln 2-\frac{\pi^2}{12}$$

4.7. Combination of Exponential Integral With Bessel Functions

1. $$\int_0^\infty E_1(ax)J_0(bx)dx=\frac{1}{b}\ln\left[\frac{b+(a^2+b^2)^{1/2}}{a}\right]$$

2. $$\int_0^\infty xE_1(ax)J_0(bx)dx=\frac{1}{b^2}\left[1-a(a^2+b^2)^{-1/2}\right]$$

3. $$\int_0^\infty x^p E_1(ax)J_{p-1}(bx)dx=\frac{1}{ab}\left(\frac{b}{2}\right)^p(a^2+b^2)^{1/2-p}\frac{\Gamma(2p)}{\Gamma(p+1)}\times{}_2F_1\left(\frac{1}{2},1;p+1;\frac{-b^2}{a^2}\right)$$
$$=\frac{1}{b}\left(\frac{b}{2}\right)^p(a^2+b^2)^{-p}\frac{\Gamma(2p)}{\Gamma(p+1)}\times{}_2F_1\left(\frac{1}{2},p;p+1;\frac{b^2}{a^2+b^2}\right) \qquad p>0$$

4. $$\int_0^\infty x^{2q+1-p}E_1(ax)J_p(bx)\,dx=\frac{1}{2}\frac{(b/2)^p}{a^{2q+2}}\times\frac{\Gamma(2q+2)}{\Gamma(p+1)}\times\frac{1}{(q+1)}$$
$$\times{}_3F_2\left(q+1,q+1,q+\frac{3}{2};q+2,p+1;\frac{-b^2}{a^2}\right) \qquad p,q>-1$$

5. $$\int_0^\infty E_1(a/x)J_1(bx)dx=\frac{2}{b}K_0(\sqrt{2ab})J_0(\sqrt{2ab})$$

6. $$\int_0^\infty E_1(a/x)\ Y_1(bx)dx=\frac{2}{b}K_0\ (\sqrt{2ab})\ Y_0\ (\sqrt{2ab})$$

7. $$\int_0^\infty E_1(x)I_0(bx)dx=\frac{1}{b}\left(\frac{\pi}{2}-\cos^{-1}b\right) \qquad 0<b\leq 1$$

8. $$\int_0^\infty E_1(a/x)\ K_1(bx)dx=\frac{2}{b}K_0\ (e^{i\pi/4}\sqrt{2ab})\ K_0\ (e^{-i\pi/4}\sqrt{2ab})$$

9. $$\int_0^\infty xE_1\ (ax)I_0(bx)dx=-\frac{1}{b^2}\left[1-a(a^2-b^2)^{-1/2}\right] \qquad a>b$$

10. $$\int_0^\infty E_1(ax)J_0(b\sqrt{x})\ dx=\frac{4}{b^2}\left[1-e^{-b^2/(4a)}\right]$$

11. $\int_0^\infty x^{1/2}E_1(ax)J_1(b\sqrt{x})\,dx=\frac{8}{b^3}\left[1-\left(1+\frac{b^2}{4a}\right)e^{-b^2/(4a)}\right]$

12. $\int_0^\infty x^{p/2}E_1(ax)J_p(b\sqrt{x})\,dx=\left(\frac{2}{b}\right)^{p+2}\gamma\left(p+1,\frac{b^2}{4a}\right)\qquad p>-1$

13. $\int_0^\infty x^{-1/2}E_1(ax)J_1(b\sqrt{x})\,dx=\frac{2}{b}\left[\gamma+\ln\left(\frac{b^2}{4a}\right)+E_1\left(\frac{b^2}{4a}\right)\right]$

14. $\int_0^\infty x^{q-p/2}E_1(ax)J_p(b\sqrt{x})\,dx=\left(\frac{b}{2}\right)^p\frac{\Gamma(q+1)}{\Gamma(p+1)}\times\frac{1}{a^{q+1}(q+1)}$
$\times{}_2F_2\left(q+1,\,q+1;\,p+1,\,q+2;\,-\frac{b^2}{4a}\right)\qquad p,\,q>-1$

15. $\int_0^\infty E_1(ax)Y_0(b\sqrt{x})\,dx=\frac{\pi}{4b^2}\left[\gamma+\ln\left(\frac{b^2}{4a}\right)-e^{-b^2/(4a)}Ei\left(\frac{b^2}{4a}\right)\right]$

16. $\int_0^\infty E_1(ax)I_0(b\sqrt{x})\,dx=\frac{4}{b^2}\left[e^{b^2/(4a)}-1\right]$

17. $\int_0^\infty x^{n/2}E_1(ax)I_n(b\sqrt{x})\,dx=(-1)^{n+1}n!\left(\frac{2}{b}\right)^{n+2}\left[1-e_n\left(-\frac{b^2}{4a}\right)e^{b^2/(4a)}\right]$

18. $\int_0^\infty x^{q-(p+1)/2}E_1(ax)\mathbf{H}_p(b\sqrt{x})\,dx=\frac{2}{a}\sqrt{\frac{2}{\pi}}\frac{b^{p+1}}{(4a)^q}\times\frac{\Gamma(q+1)\Gamma(q+1)}{\Gamma\left(p+\frac{3}{2}\right)\Gamma(q-p+2)}$
$\times{}_3F_3\left(1,\,q+1,\,q+1;\,\frac{3}{2},\,p+\frac{3}{2},\,q+2;\,\frac{-b^2}{4a}\right)$
$q>-1,\,p>-\frac{3}{2}$

19. $\int_0^\infty x^{p/2}e^{ax}E_1(ax)J_p(b\sqrt{x})\,dx=\frac{2}{b}\frac{\Gamma(p+1)}{a^{(p+1)/2}}e^{b^2/(8a)}W_{-(p+1)/2,\,p/2}\left(\frac{b^2}{4a}\right)\qquad -1<p<\frac{1}{2}$

20. $\int_0^\infty x^{(p+1)/2}e^xE_1(x)Y_p(b\sqrt{x})\,dx=\frac{\sqrt{\pi}}{b}\Gamma\left(p+\frac{3}{2}\right)e^{b^2/8}\times W_{-(p+2)/2,\,p/2}(b^2/4)$
$-\frac{3}{2}<p<\frac{1}{2}$

21. $\int_0^\infty x^{(p+3)/2}e^xE_1(x)\,Y_p(b\sqrt{x})\,dx=\frac{-3\sqrt{\pi}}{2b}\Gamma\left(p+\frac{5}{2}\right)e^{b^2/8}\times W_{-(p+4)/2,\,p/2}(b^2/4)$
$-\frac{5}{2}<p<-\frac{3}{2}$

22. $\int_0^\infty e^{ax}E_1(ax)\mathbf{H}_0(b\sqrt{x})dx=\frac{\pi}{a}e^{b^2/(4a)}\operatorname{erfc}\left(\frac{b}{2\sqrt{a}}\right)$

23. $\int_0^\infty x^{-1/2}e^{x(1-a/2)}K_0\left(\frac{1}{2}ax\right)E_1(x)dx=\left(\frac{\pi^5}{a}\right)^{1/2}{}_2F_1\left(\frac{1}{2},\frac{1}{2};\,1;\,1-\frac{1}{a}\right)$

24. $\int_0^\infty e^{-bx/2}J_0(bx/2)E_1(ax)dx=\frac{\sqrt{2}}{b}\ln\left[\frac{(a+b)+\sqrt{(a+b)^2+a^2}}{a(1+\sqrt{2})}\right]$

25. $$\int_0^\infty e^{\pm bx/2} I_0\left(\frac{bx}{2}\right) E_1(ax)\,dx = \frac{2}{\sqrt{a}(\sqrt{a}+\sqrt{a\mp b})} \qquad a > b \text{ for upper sign}$$

26. $$\int_0^\infty x^p e^{x/2} I_p\left(\frac{x}{2}\right) E_1(x)\,dx = \frac{\pi^{-1/2}}{p+\frac{1}{2}}\,\Gamma(1+2p)\Gamma\left(\frac{1}{2}-p\right) \qquad -\frac{1}{2} < p < \frac{1}{2}$$

27. $$\int_0^\infty x^p e^{(a-1)x} I_p(x) E_1(ax)\,dx = \frac{\pi^{-1/2}}{a}\left(\frac{2}{a^2}\right)^p \frac{\Gamma(1+2p)\Gamma\left(\frac{1}{2}-p\right)}{p+\frac{1}{2}}$$
$$\times {}_2F_1\left(p+\frac{1}{2},\, 2p+1;\, p+\frac{3}{2};\, 1-\frac{2}{a}\right) \qquad a \geq 1, -\frac{1}{2} < p < \frac{1}{2}$$

28. $$\int_0^\infty x E_1[a\{b+(b^2+x^2)^{1/2}\}] J_0(cx)\,dx = \frac{\exp\{-b(a+\sqrt{a^2+c^2})\}}{(a+\sqrt{a^2+c^2})\sqrt{a^2+c^2}}$$

29. $$\int_0^\infty E_1(ax) J_0(b\sqrt{x}) \ln x\,dx = \frac{4}{b^2} e^{-b^2/(4a)}\left\{Ei\left(\frac{b^2}{4a}\right) + \ln a - \ln\left(\frac{b^2}{4a}\right)\right\}$$
$$-\frac{8}{b^2}\left\{E_1\left(\frac{b^2}{4a}\right) + \ln a + \ln\left(\frac{b^2}{4a}\right)\right\} + \frac{4}{b^2}(\ln a - 3\gamma)$$

30. $$\int_0^\infty x^p E_1(x) J_{\lambda+\nu}(a\sqrt{x}) J_{\lambda-\nu}(a\sqrt{x})\,dx = \frac{(a/2)^{2\lambda}\Gamma(p+\lambda+1)}{\Gamma(\lambda+\nu+1)\Gamma(\lambda-\nu+1)(p+\lambda+1)}$$
$$\times {}_4F_4(\lambda+\tfrac{1}{2},\, \lambda+1,\, \lambda+p+1,$$
$$\lambda+p+1;\, \lambda-\nu+1,\, \lambda+\nu+1,$$
$$2\lambda+1,\, \lambda+p+2;\, -a^2) \qquad \lambda+p+1 > 0$$

31. $$\int_0^\infty E_1(x/a)\,\text{ber}\,(2\sqrt{x})\,dx = \sin a$$

32. $$\int_0^\infty E_1(x/a)\,\text{bei}\,(2\sqrt{x})\,dx = (1-\cos a)$$

4.8. Combination of Exponential Integral With Other Special Functions

1. $$\int_0^\infty E_1(ax)\gamma(p+1,\, bx)\,dx = \frac{1}{a}\frac{\Gamma(p+1)}{\left(1+\frac{a}{b}\right)^{p+1}} - \frac{1}{b}\int_0^\infty e^{-t} t^{(p+1)} E_1\left(\frac{a}{b}t\right) dt \qquad p > -1$$

See 4.2.20 for Evaluation of this integral.

2. $$\int_0^\infty E_1(ax)\Gamma(p+1,\, bx)\,dx = \frac{1}{a}\Gamma(p+1)\left[1-\left(1+\frac{a}{b}\right)^{-p-1}\right]$$
$$+\frac{1}{b}\int_0^\infty e^{-t} t^{p+1} E_1(at/b)\,dt \qquad p > -1$$

See 4.2.20 for Evaluation of this integral.

3. $$\int_0^\infty E_1(ax)\gamma(p+1, bx)e^{bx}dx=\frac{1}{b}\Gamma(p+1)\sum_{m=0}^{\infty}\frac{1}{(m+p+2)}\left(\frac{b}{a}\right)^{m+p+2}$$
$$=\frac{1}{b}\Gamma(p+1)\left(\frac{b}{a}\right)^{p+2}\Phi\left(\frac{b}{a}, 1, p+2\right) \qquad p>-1, a>b$$

4. $$\int_0^\infty E_1(ax)\gamma(n+1, bx)e^{bx}dx=\frac{-n!}{b}\left[\ln\left(\frac{a-b}{b}\right)+\sum_{m=1}^{n-1}\frac{1}{m}\left(\frac{b}{a}\right)^m\right] \qquad a>b$$

5. $$\int_0^\infty x^{p-1}e^{(b-1)x}{}_1F_1(a; p; x)E_1(bx)dx=\frac{\Gamma(p)\Gamma(p)\Gamma(1-a)}{b^p\Gamma(p+1-a)}$$
$$\times{}_2F_1\left(p-a, p; p-a+1; 1-\frac{1}{b}\right) \qquad p>0, a<1$$

6. $$\int_0^\infty x^p{}_1F_1(2p+1-a; 2p+1; x)J_{2p}(2\sqrt{bx})E_1(x)dx$$
$$=\frac{\Gamma(2p+1)}{\Gamma(a)}\cdot b^{p-a}e^b\Gamma(a-2p, b)\gamma(a, b) \qquad p>-\frac{1}{2}, a>-1$$

7. $$\int_0^\infty x^{p-1/2}e^{-ax/2}M_{\kappa,\mu}(ax)E_1(bx)dx=\frac{a^{\mu+1/2}}{b^{\mu+p+1}}\frac{\Gamma(p+\mu+1)}{(p+\mu+1)}$$
$$\times{}_3F_2\left(\frac{1}{2}+\kappa+\mu, p+\mu+1, p+\mu+1; 2\mu+1, p+\mu+2; -\frac{a}{b}\right) \qquad p+\mu>-1$$

8. $$\int_0^\infty e^{-(p-1)x}L_n(px)E_1(x)dx=\frac{1}{(n+1)}{}_2F_1(1, n+1; n+2; 1-p) \qquad 0<p<2$$

9. $$\int_0^\infty L_n(x)E_1(ax)dx=\frac{1}{(n+1)}\left[1-\left(1-\frac{1}{a}\right)^{n+1}\right]$$

10. $$\int_0^\infty si(bx)E_1(ax)dx=-\frac{1}{a}\tan^{-1}\left(\frac{a}{b}\right)-\frac{1}{2b}\ln\left(1+\frac{b^2}{a^2}\right)$$

11. $$\int_0^\infty Ci(ax)E_1(ax)dx=-\frac{1}{4a}(\pi+2\ln 2)$$

12. $$\int_0^\infty \operatorname{erf}(\sqrt{bx})E_1(ax)dx=\frac{1}{a}\left(1+\frac{a}{b}\right)^{1/2}+\frac{1}{2b}\ln\left(\frac{\sqrt{a+b}-\sqrt{b}}{\sqrt{a+b}+\sqrt{b}}\right)$$

13. $$\int_0^\infty \operatorname{erfc}\left(\frac{b}{2\sqrt{x}}\right)E_1(a^2x)dx=\frac{1}{a^2}(1-ab)e^{-ab}+b^2E_1(ab)$$

14. $$\int_0^\infty \operatorname{erfc}(ax)E_1\left(\frac{b^2}{x^2}\right)\frac{dx}{x^3}=\frac{1}{2b^2}(1-2ab)e^{-2ab}+2a^2E_1(2ab)$$

15. $$\int_0^\infty \operatorname{erfc}(ax)E_1\left(\frac{b^2}{x^2}\right)\frac{dx}{x}=[\zeta(2)+(\gamma+\ln 2ab)^2]+2\sum_{m=1}^{\infty}\frac{(-2ab)^m}{m!m^2}$$

4.9. Miscellaneous Integrals

1. $$\int_0^\infty (1-be^{-px})^{-1}E_1(ax)dx=\frac{1}{a}+\frac{1}{p}\sum_{m=1}^{\infty}\frac{b^m}{m}\ln\left(1+\frac{mp}{a}\right) \qquad -1\leq b<1$$

2. $\int_0^\infty (1+be^{px})^{-1}E_1(ax)dx=-\frac{1}{p}\sum_{m=1}^{\infty}\frac{(-1)^m}{b^m m}\ln\left(1+\frac{mp}{a}\right) \qquad b>1$

3. $\int_0^\infty \operatorname{sech} xE_1(ax)dx=2\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)}\ln\left(1+\frac{2m+1}{a}\right)$

4. $\int_0^\infty \tanh xE_1(ax)dx=\frac{1}{a}+\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\ln\left(1+\frac{2m}{a}\right)$

5. $\int_0^\infty \ln(\cosh x)E_1(ax)dx=\frac{3}{2a^2}-\frac{2}{a}\ln 2-\sum_{m=1}^{\infty}\frac{(-1)^m}{m^2}\ln\left(1+\frac{2m}{a}\right)$

In 6–8, let $F(p,q)=\int_{2\sqrt{ab}}^\infty K_q(x)\frac{dx}{x^p}$

6. $\int_0^\infty x^{p-1}e^{-b/x}E_1(ax)dx=4(2b)^pF(p+1,p)$

7. $\int_0^\infty \Gamma(p,b/x)E_1(ax)dx=b2^{4-p}F(3-p,p)$

8. $\int_0^\infty E_1(b/x)E_1(ax)dx=16bF(3,0)$

In 9–12, let $G(p,q)=\int_{ab}^\infty e^xK_q(x)\frac{dx}{x^p}$

9. $\int_0^\infty [x(x+2b)]^{p-1/2}E_1(ax)dx=\frac{(2b^2)^p}{\sqrt{\pi}}\Gamma\left(p+\frac{1}{2}\right)G(p+1,p) \qquad p>-\frac{1}{2}$

10. $\int_0^\infty \frac{(x+b)}{\sqrt{x(x+2b)}}E_1(ax)dx=bG(1,1)$

11. $\int_0^\infty [(\sqrt{x+2b}+\sqrt{x})^{2p}-(\sqrt{x+2b}-\sqrt{x})^{2p}]E_1(ax)dx=p(2b)^{p+1}G(2,p) \qquad p>0$

12. $\int_0^\infty [x(x+2b)]^{-p/2}P_q^p\left(1+\frac{x}{b}\right)E_1(ax)dx=\sqrt{\frac{2}{\pi}}b^{1-p}G\left(\frac{3}{2}-p,\frac{1}{2}+q\right)$

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Principal Ideals in Matrix Rings

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It is shown that every left ideal of the complete matrix ring of a given order over a principal ideal ring is principal, and a partial converse is proven.

Key words: Dedekind ring; matrix ring; non-Noetherian ring; principal ideal ring.

1. Introduction

Let R be a ring with a unity 1, and let n be a positive integer. It is well-known [3, p. 37][1] that every two-sided ideal of R_n (the complete matrix ring of order n over R) is necessarily of the form M_n, where M is a two-sided ideal of R. Simple examples show that this result no longer holds for one-sided ideals. In this note we investigate the left ideals of R_n in the case when R is a principal ideal ring (an integral domain in which every ideal is principal). We shall prove

THEOREM 1: *If* R *is a principal ideal ring, then every left ideal of* R_n *is principal.*

The proof of Theorem 1 depends upon the fact that if A is any $p \times q$ matrix over R, then a unit matrix U of R_p exists such that the $p \times q$ matrix UA is upper triangular [2, p. 32].

We also establish the following partial converse to Theorem 1:

THEOREM 2: *If* R *is not Noetherian or if* R *is a Dedekind ring but not a principal ideal ring, then* R_n *contains a nonprincipal left ideal.*

For general information on rings, see [3]. For information on Dedekind rings, see [1, p. 101].

2. Proofs

We denote the matrix of R_n which has 1 in position (i, j) and 0 elsewhere by E_{ij}. We first prove

LEMMA 1: *Suppose that every left ideal of* R *has a finite* R*-basis. Then so has every left ideal of* R_n.

PROOF: Let a be a left ideal of R_n. Let a_k, $2 \leq k \leq n$, be the subset of a consisting of all matrices of a whose first $k-1$ columns are 0; and set $a_1 = a$. Then, as is easily verified, a_k is a left ideal of R_n for $1 \leq k \leq n$.

Let M_{ik}, $1 \leq i \leq n$, be the set of elements of R occurring in the (i, k) position of all matrices of a_k, $1 \leq k \leq n$. Then M_{ik} is a left ideal of R (since a_k is a left ideal of R_n) and so has a finite R-basis, say

$$m_{ik}^l,\ 1 \leq l \leq l_{ik}.$$

Hence we can find l_{ik} matrices of a_k, say A_{ik}^l, such that the (i, k)th entry of A_{ik}^l is m_{ik}^l. It follows that the l_{ik} matrices

$$B_{ik}^l = E_{ii}A_{ik}^l,\ 1 \leq i, k \leq n,\ 1 \leq l \leq l_{ik},$$

also belong to a_k, have m_{ik}^l as their (i, k)th entry, but have nonzero entries in the ith row only. These

[1] Figures in brackets indicate the literature at the end of this paper.

matrices constitute a finite R-basis for a. For suppose that A is any element of a. We first find elements r_{i1}^l of R such that

$$A-\sum_{i=1}^{n}\sum_{l=1}^{l_{i1}} r_{i1}^l B_{i1}^l = A_2 \epsilon a_2;$$

we then find elements r_{i2}^l of R such that

$$A_2-\sum_{i=1}^{n}\sum_{l=1}^{l_{i2}} r_{i2}^l B_{i2}^l = A_3 \epsilon a_3;$$

and continuing in this manner, we determine elements r_{ik}^l of R such that

$$A=\sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{l=1}^{l_{ik}} r_{ik}^l B_{ik}^l.$$

This completes the proof.

We now prove Theorem 1. Let a be a left ideal of R_n. By Lemma 1, a possesses a finite R-basis, say $B_1, B_2, \ldots, B_t$. Let B be the $nt \times t$ matrix

$$B=\begin{pmatrix} B_1 \\ B_2 \\ \cdot \\ \cdot \\ \cdot \\ B_t \end{pmatrix}$$

Let U be a unit matrix of R_{nt} such that $UB=T$ is upper triangular. Thus

$$UB=T=\begin{bmatrix} H \\ 0 \end{bmatrix},$$

where H is an $n \times n$ upper triangular matrix, and the zero block 0 is $(nt-n) \times n$. We shall show that $a=R_nH$. For if we write $U=(U_{ij})$, where the matrices U_{ij} are $n \times n$, then

$$\sum_{j=1}^{t} U_{1j}B_j = H,$$

so that $H \epsilon a$, implying that

$$R_nH \subset a.$$

If we then write $U^{-1}=V=(V_{ij})$, where the matrices V_{ij} are $n \times n$, then V belongs to R_{nt} (since U is a unit matrix of R_{nt}) and from $B=VT$ we find that

$$B_i = V_{i1}H, \qquad 1 \leq i \leq t,$$

implying that

$$a \subset R_nH.$$

This completes the proof of Theorem 1.

To prove Theorem 2, we first observe that for any left ideal M of R, the left ideal M_n of R_n can be principal only if M has a set of n or fewer generators. In particular, if R is non-Noetherian, M can be chosen to violate this condition.

We now assume that R is a Dedekind ring and that any ideal in R can be generated by at most n elements. Let S be a nonprincial ideal in R, and let $\mathscr{S}$ be the subset of R_n consisting of all matrices with first column entries in S and all other entries arbitrary members of R. Clearly, $\mathscr{S}$ is a left ideal in R_n. We shall show that $\mathscr{S}$ is not principal.

Suppose the contrary. Let $X=(x_{ij})$ generate $\mathscr{S}$, so that $\mathscr{S}=R_nX$. Clearly the x_{i1} generate S; $S=\{x_{11}, x_{21}, \ldots, x_{n1}\}$. We may assume that x_{11} is not zero, since we may interchange the rows of X by left multiplication by a permutation matrix. Let $d=\det X$. Since $\mathscr{S}$ contains nonsingular matrices (for example, diag $(x_{11}, 1, \ldots, 1)$) Xmust be nonsingular and thus d is a nonzero element of S. Let $Y=(y_{ij})$ be the adjoint of X, so that

$$XY=YX=dI.$$

Then $Y \epsilon R_n$, and if C is any matrix in $\mathscr{S}$, every element of CY must be divisible by d. First choose $C = x_{i1} E_{11}$, $1 \leq i \leq n$. We obtain

$$\{d\} | \{x_{i1} y_{1j}\}, \qquad 1 \leq i, j \leq n. \tag{1}$$

Next choose $C = E_{1j}$, $2 \leq i \leq n$. We obtain

$$\{d\} | \{y_{1j}\}, \qquad 2 \leq i \leq n, \qquad 1 \leq j \leq n. \tag{2}$$

Put $y = \{y_{11}, y_{12}, \ldots, y_{1n}\}$. Then (2) implies that $y\{d\}^{n-1} | \{\det Y\}$; and since $\det Y = d^{n-1}$, $y = \{1\} = R$. Hence $\{y_{11}, y_{12}, \ldots, y_{1n}\} = \{1\} = R$. But now (1) and (2) imply that $\{d\} | \{x_{i1}\}$, $1 \leq i \leq n$. Write

$$x_{i1} = \beta_i d, \qquad \beta_i \epsilon R, \qquad 1 \leq i \leq n. \tag{3}$$

Since $d \epsilon S$ and the x_{i1} are a basis for S, elements r_i of R exist such that

$$d = \sum_{i=1}^{n} r_i x_{i1}.$$

But now (3) implies that

$$\sum_{i=1}^{n} r_i \beta_i = 1,$$

and hence

$$S = \{x_{11}, x_{21}, \ldots, x_{n1}\} = \{\beta_1, \beta_2, \ldots, \beta_n\}\{d\} = \{d\}.$$

Thus S is principal, a contradiction. This completes the proof of Theorem 2.

3. References

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The Second Orthogonality Conditions in the Theory of Proper and Improper Rotations IV. Solution of the Trace and Secular Equations*

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The equation which connects the trace of a rotation matrix and that of its square, and the secular equation for a rotation matrix, both of which are direct results of the second orthogonality conditions, are solved by purely analytic methods based on the group property and the periodicity property of rotation matrices. The point is thus made that the well known formulas for the trace of a rotation matrix in terms of the angle of rotation and for the rotation matrix itself in terms of the axis and angle of rotation, are closely related to the algebraic properties of rotation matrices.

Key words: Closed form of rotation matrix; orthogonal transformation; rotation; rotation matrix; secular equation; trace equation; trace formula.

1. Introduction

In paper I [1][1] of this series two of the results derived from the second orthogonality conditions were an equation which connects the trace of an orthogonal transformation matrix with that of its square, and the secular equation for a transformation matrix. It was stated that for rigid rotations these equations could be solved directly by analytic methods based only on very general properties of rotation matrices. The details of the solutions were deferred to the present paper.

For completeness we rewrite here the equations to be solved in this paper and the solutions, using R to denote a rotation matrix. The equation which connects $tr\,R$ and $tr\,(R^2)$ is eq (I 15) with $p'/p=1$,

$$(tr\,R)^2-tr\,(R^2)=2\,tr\,R. \tag{1}$$

The secular equation for R is eq (I 22) with $p'/p=1$,

$$R^3-(tr\,R)\,R^2+(tr\,R)\,R-I=0, \tag{2}$$

where I is the identity matrix. The well known expression for tr R in terms of the rotation angle α is

$$tr\,R=1+2\,cos\,\alpha. \tag{3}$$

When this is inserted in eq (2), the secular equation becomes

$$R^3-(1+2\cos\alpha)\,R^2+(1+2\cos\alpha)R-I=0. \tag{4}$$

The solution of eq (4) is

$$R(\alpha)=I-N\sin\alpha+N^2(1-\cos\alpha), \tag{5}$$

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[1] Figures in brackets indicate the literature references at the end of this paper.

where N is given in terms of the components of the axis of rotation by eqs (I 26) and (I 27). The end results of this paper are derivations of eq (3) from eq (1) and eq (5) from eq (4).

The properties of rotation matrices which the derivations in this paper are based on are the group property and the periodicity property. The group property follows from the fact that the product of two rotation matrices is a matrix which has determinant unity and satisfies the first orthogonality conditions, and therefore represents a rotation about some axis. The computation of the axis and angle of the product rotation was treated as an example in section 4 of paper II [2]. For the purposes of this paper a restricted statement of the group property suffices. This is the statement that the product of two matrices representing rotations about the same axis is a matrix representing a rotation about that axis by an angle which is the sum of the angles in the component rotations. If $R(\alpha)$ represents a rotation about a fixed axis by an angle α, the restricted statement of the group property is

$$R(\alpha_2)R(\alpha_1)=R(\alpha_1+\alpha_2). \tag{6}$$

The second property of rotation matrices that we use is the periodicity property. We take as the statement of this property the equality of $R(2\pi)$ (for any axis) with $R(0)$, which is the identity matrix,

$$R(2\pi)=R(0)=I, \tag{7}$$

together with the requirement that for no angle smaller than 2π is $R(\alpha)$ equal to the identity. The second part of the statement is necessary in order that α be the true angle of rotation rather than some multiple of it. From eqs (6) and (7) an alternative form of the periodicity property is

$$R(\alpha+2\pi)=R(\alpha)R(2\pi)=R(\alpha)\cdot \tag{8}$$

The solutions eqs (3) and (5) are well known and are not the objects of interest in this paper. The objects of interest are the derivations of these solutions to be presented in sections 2 and 3 respectively. The aim in giving these new derivations is to call attention to the fact that the trace formula eq (3) and the closed form eq (5) are direct consequences of algebraic properties of rotation matrices. Thus, although the derivations given here do not give much geometrical insight into the nature of the solutions, they may give a deeper insight.

2. Solution of the Trace Equation

We represent the dependence[2] of $tr\,R$ on α by the function $g(\alpha)$,

$$g(\alpha)\equiv trR(\alpha)\cdot \tag{9}$$

We now observe from eq (6) that $[R(\alpha)]^2=R(2\alpha)$. It then follows from eq (9) that

$$tr[R(\alpha)]^2=trR(2\alpha)=g(2\alpha)\cdot \tag{10}$$

We may therefore rewrite eq (1) as

$$[g(\alpha)]^2-g(2\alpha)=2g(\alpha), \tag{11}$$

which is a functional equation for the unknown function $g(\alpha)$. A more transparent functional equation is obtained by defining the function

$$h(\alpha)\equiv\tfrac{1}{2}[g(\alpha)-1], \tag{12}$$

and rewriting eq (11) in terms of h. The functional equation for h can be written

$$[h(\alpha)]^2=\tfrac{1}{2}[h(2\alpha)+1]. \tag{13}$$

Equations (11) and (13) are both independent of the axis of rotation.

[2] The fact that $g(\alpha)$ is independent of the axis of rotation is not immediately apparent from eq (1). It will become apparent as the derivation proceeds.

From eq (13) one can begin to see the connection between the result $h(\alpha)=\cos\alpha$, and eq (1). The structure of eq (13) is identical to that of the formula for the cosine of a double angle. However, $\cos\alpha$ is not the only solution to eq (13). We will calculate all of its solutions and then see how the geometrically meaningful solution is selected by the periodicity condition.

The coordinate transformations which underlie rotation matrices are analytic functions of the parameters which specify a rotation.[3] It follows from this that $h(\alpha)$ can be expanded in a convergent real Taylor series about $\alpha=0$. Now we observe from eq (13) that if $h(\alpha)$ is a solution, so also is $h(-\alpha)$. That is, $h(\alpha)$ is an even function of α, and when expanded in a power series about $\alpha=0$ contains only even powers of α. We therefore write the assumed series in the form

$$h(\alpha)=\sum_{m=0}^{\infty} C_{2m}\alpha^{2m}, \tag{14}$$

where the C_j are constant expansion coefficients. The square of this series is

$$[h(\alpha)]^2=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{2m}C_{2n}\alpha^{2(m+n)}. \tag{15}$$

We notice that in h^2, α occurs raised to the power $2(m+n)$. This suggests that we change the variables of summation in eq (15) from m and n, to a pair of variables one of which is $m+n$. Accordingly we define new variables j and k by $j\equiv m+n$, $k\equiv m-n$. To find the ranges of j and k we note that j is nonnegative and increases in steps of one. Hence its range is all of the nonnegative integers. We next observe that for a given value of j, there are $j+1$ allowed pairs of values of m and n in which the sum of the values of m and n in each pair equals the given j value. These pairs (m, n) are $(0, j)$, $(1, j-1)$, $(2, j-2)$, . . ., $(j-1, 1)$, $(j, 0)$. The values of k corresponding to these pairs are $k=-j, -j+2, -j+4$, . . ., $j-2, j$. We see that k ranges from $-j$ to $+j$ in steps of two. It is convenient to parametirize k as $k=-j+2r$ where r is an integer taking on the values $0, 1, 2$, . . ., j. Since the subscripts on the C's in eq (15) are $2m$ and $2n$, we have to express these in terms of j and r. The results are $2m=2r$, $2n=2(j-r)$. The series for h^2 may then be written

$$[h(\alpha)]^2=\sum_{j=0}^{\infty}\left[\sum_{r=0}^{j} C_{2r}C_{2(j-r)}\right]\alpha^{2j}. \tag{16}$$

Note that the order of summation in eq (16) cannot be interchanged since the range of r depends upon j.

If we insert the series eq (14) evaluated for argument 2α, and the series eq (16) for argument α into the functional equation (13), we get

$$\sum_{j=0}^{\infty}\left[\sum_{r=0}^{j} C_{2r}C_{2(j-r)}\right]\alpha^{2j}=\frac{1}{2}\left[\sum_{j=0}^{\infty} 2^{2j}C_{2j}\alpha^{2j}+1\right]. \tag{17}$$

By equating the coefficients of like powers of α on the two sides of this equation we arrive at a set of equations for the determination of the C_{2j}. We break this process into the cases $j=0$, $j=1$, and $j\geq 2$.

(a) $j=0$. This yields an equation for C_0,

$$C_0^2=\tfrac{1}{2}(C_0+1), \tag{18}$$

whose solutions are $C_0=-\frac{1}{2}$, and $C_0=1$. The former solution can be shown to generate a "series" all of whose higher order coefficients are zero. This corresponds to the constant solution $h(\alpha)=-\frac{1}{2}$ of eq (13), or by eq (12) to the trivial solution $g(\alpha)=0$ of eq (11). This solution can be ruled out because it has arbitrarily small periods in α, whereas the second part of the periodicity condition requires that there be no period smaller than 2π. We therefore concentrate on the series generated by the value $C_0=1$.

(b) $j=1$. Equating coefficients of α^2 on both sides of eq (17) we get

$$C_0C_2=C_2, \tag{19}$$

[3] This is equivalent to saying that the rotation group is a Lie group.

and setting $C_0=1$ we get the identity $C_2=C_2$. This means that C_2 is not determined by eq (13) and is therefore a free parameter of the solution. All of the suceeding coefficients will depend on the value of C_2.

(c) $j \geq 2$. Equating the coefficients of α^{2j} on both sides of eq (17) we have

$$\sum_{r=0}^{j} C_{2r}C_{2(j-r)}=2^{2j-1}C_{2j}. \tag{20}$$

Now since $j \geq 2$ we can write the first and last terms (i.e., those for $r=0$ and $r=j$) on the left side of eq (20) separately from the rest of the sum, giving

$$2C_0C_{2j}+\sum_{r=1}^{j-1} C_{2r}C_{2(j-r)}=2^{2j-1}C_{2j}. \tag{21}$$

Upon setting $C_0=1$ this equation can be solved for C_{2j} to give

$$C_{2j}=\frac{1}{2[2^{2(j-1)}-1]}\sum_{r=1}^{j-1} C_{2r}C_{2(j-r)}. \tag{22}$$

This is a recursion relation for C_{2j}.

Computation of the first few coefficients from eq (22) gives

$$C_4=\frac{C_2^2}{6}=\frac{(2C_2)^2}{4!},$$

$$C_6=\frac{C_2C_4}{15}=\frac{C_2^3}{90}=\frac{(2C_2)^3}{720}=\frac{(2C_2)^3}{6!},$$

$$C_8=\frac{2C_2C_6+C_4^2}{2(63)}=\frac{(2C_2)^4}{40{,}320}=\frac{(2C_2)^4}{8!}.$$

This suggests the assumption that

$$C_{2j}=\frac{(2C_2)^j}{(2j)!}, \tag{23}$$

for all $j \geq 0$, since it is automatically true for $j=0$ and 1, and has just been demonstrated to be true for $j=2$, 3, and 4. To prove it inductively for all j, we assume that its truth has been demonstrated for all j up to a certain value, say $j=N$, where N is an integer equal to or greater than two. Then we use eq (22) to compute C_{2j} for $j=N+1$. Since all of the subscripts on the C's on the right side of eq (22) are less than or equal to N we can use the assumption eq (23) for those C's with the result

$$C_{2(N+1)}=\frac{(2C_2)^{N+1}}{2(2^{2N}-1)}\sum_{r=1}^{N}\frac{1}{(2r)![2(N-r+1)]!}. \tag{24}$$

The sum on the right side of this equation can be evaluated by using the property of the binomial coefficients that (ref. [3])

$$\sum_{r=0}^{N}\binom{2N}{2r}=2^{2N-1}. \tag{25}$$

From this property it follows by algebraic manipulation that

$$\sum_{r=1}^{N}\frac{1}{(2r)![2(N-r+1)]!}=\frac{2(2^{2N}-1)}{[2(N+1)]!}. \tag{26}$$

Insertion of this value for the sum on the right side of eq (24) gives

$$C_{2(N+1)}=\frac{(2C_2)^{N+1}}{[2(N+1)]!},$$

which is just eq (23) for $j=N+1$. This then furnishes the inductive step that allows us to assert the truth of eq. (23) for all j from its demonstrated truth for $j \leq N$.

If we now set $2C_2=K^2$ (where K may be real or pure imaginary, depending on whether C_2 is positive or negative) and insert eq (23) for the coefficients into the series eq (14), we can recognize the resulting series as the Taylor expansion about $\alpha=0$ of cosh $K\alpha$,

$$h(\alpha)=\cosh K\alpha. \tag{27}$$

Equation (27) is therefore the most general, regular solution of the "double angle cosine formula" eq (13). The lack of uniqueness of the solution is expressed by the arbitrariness of the parameter K.

From the definition eq (12) we have

$$g(\alpha)=1+2\cosh K\alpha. \tag{28}$$

Since, from the first part of the periodicity property each element of $R(\alpha)$ has period 2π, so does $g(\alpha)$. This restricts K to be the imaginary number mi where m is any positive or negative integer. The second part of the periodicity property, which requires that $g(\alpha)$ have no period smaller than 2π, selects the particular values $m=\pm 1$. The result for either value of m is

$$g(\alpha)=tr\ R(\alpha)=1+2\cos\alpha,$$

which is eq (3).

3. Solution of the Secular Equation

The group property (6) allows us to replace $[R(\alpha)]^3$ and $[R(\alpha)]^2$ in eq (4) by $R(3\alpha)$ and $R(2\alpha)$. respectively. The resulting equation contains no rotation matrix raised to a power higher than the first.

$$R(3\alpha)-(1+2\cos\alpha)R(2\alpha)+(1+2\cos\alpha)R(\alpha)-I=0. \tag{29}$$

We can therefore try to solve this equation by a Fourier series between 0 and 2π in α, rather than by a power series as was done with the trace equation. The advantage of such a Fourier series solution is that it automatically incorporates the periodicity property.

It turns out to be more practical for this problem to use a Fourier series for $R(\alpha)$ in the complex exponentials $e^{im\alpha}$ rather than in sin $m\alpha$ and cos $m\alpha$. We write this series as

$$R(\alpha)=\sum_{m=-\infty}^{\infty}\langle m|R\rangle\, e^{im\alpha}. \tag{30}$$

In eq (30) we are using Dirac notation [4] for the Fourier coefficient matrices in order to avoid complicated expressions in subscripts, in equations for the coefficients to be derived soon. The Fourier coefficients are given by the usual formula

$$\langle m|R\rangle=\frac{1}{2\pi}\int_0^{2\pi}R(\alpha)e^{-im\alpha}\,d\alpha. \tag{31}$$

The entire dependence of $R(\alpha)$ on the axis of rotation is contained in these Fourier coefficients.

We now substitute the series eq (30) evaluated for the appropriate arguments into each of the terms of eq (29). We also express the coefficients $1+2\cos\alpha$ as $1+e^{i\alpha}+e^{-i\alpha}$. The resulting equation can be written

$$\sum_{m=-\infty}^{\infty}\langle m|R\rangle\,[e^{3im\alpha}-e^{2im\alpha}+e^{im\alpha}-e^{i(2m+1)\alpha}+e^{i(m+1)\alpha}-e^{i(2m-1)\alpha}+e^{i(m-1)\alpha}]=I. \tag{32}$$

Equations for the Fourier coefficients are extracted from eq (32) by the use of the orthonormality property of the complex exponentials

$$\int_0^{2\pi} e^{i(r-k)\alpha}\, d\alpha = 2\pi\delta_{rk}, \tag{33}$$

where δ_{rk} is the Kronecker delta, equal to one if $r=k$ and equal to zero if $r\neq k$. Multiplication of eq (32) by $e^{-ik\alpha}$, and integration of the resulting equation with respect to α from 0 to 2π yields, with the help of eq (33)

$$\sum_{m=-\infty}^{\infty}\langle m|R\rangle\,[\delta_{k,3m}-\delta_{k,2m}+\delta_{k,m}-\delta_{k,2m+1}+\delta_{k,m+1}-\delta_{k,2m-1}+\delta_{k,m-1}]=\delta_{k0}I. \tag{34}$$

In eq (34) k is any integer, positive, negative, or zero.

We divide the consideration of eq (34) into three separate cases based on the value of k.

(a) $k=0$.

For this case $\delta_{k0}=1$. On the left side of eq (34), the first, second, and third terms contribute for $m=0$. The fifth term contributes for $m=-1$ while the seventh term contributes for $m=1$. The fourth and sixth terms do not contribute for any value of m. The equation then reduces to

$$\langle 0|R\rangle+\langle 1|R\rangle+\langle -1|R\rangle=I. \tag{35}$$

(b) k is a positive or negative *even* integer, but not zero.

We represent k by $2j$ where j is any positive or negative integer. For this case $\delta_{k0}=0$. On the left side of eq (34) the fourth and sixth terms do not contribute for any value of m. The relation between the coefficients for this case is

$$\langle \tfrac{3}{2}j|R\rangle-\langle j|R\rangle+\langle 2j|R\rangle+\langle 2j-1|R\rangle+\langle 2j+1|R\rangle=0. \tag{36}$$

(c) k is a positive or negative *odd* integer.

We represent k by $2j+1$ where j is a positive or negative integer, or zero. We again have $\delta_{k0}=0$ in eq (34). On the left side of that equation the sole noncontributing term is the second one. The relation between the coefficients which follows from eq (34) for this case is

$$\langle \tfrac{1}{2}(2j+1)|R\rangle+\langle 2j+1|R\rangle-\langle j|R\rangle+\langle 2j|R\rangle-\langle j+1|R\rangle+\langle 2(j+1)|R\rangle=0. \tag{37}$$

Evaluation of the coefficients proceeds by setting j equal to the first few of its allowed values in eqs (36) and (37), and trying to find a general trend which can then be tested inductively. In this process we set equal to zero any coefficient $\langle m|R\rangle$ in which m is not an integer. Thus, setting $j=0$ in eq (37) and equating $\langle \frac{1}{2}|R\rangle$ to zero, we find that $\langle 2|R\rangle=0$. Setting $j=1$ in eqs (36) and (37), leads to the results $\langle 3|R\rangle=0$, $\langle 4|R\rangle=0$. The value $j=-1$ leads to $\langle -2|R\rangle=0$, $\langle -3|R\rangle=0$. The value of $j=2$ leads to $\langle 5|R\rangle=0$, $\langle 6|R\rangle=0$, while $j=-2$ leads to $\langle -4|R\rangle=0$, $\langle -5|R\rangle=0$. The results $\langle 7|R\rangle=\langle 8|R\rangle=\langle -6|R\rangle=\langle -7|R\rangle=0$ follow from setting $j=3$ and then -3.

These results suggest that $\langle m|R\rangle=0$ for all $|m|\geq 2$. To test this we follow the trend suggested by these results and assume it has been shown from eqs (36) and (37) for all integer values of $|j|$ up to and including some value J greater than one, that

$$\langle 2|R\rangle=\langle 3|R\rangle=\ \ldots\ =\langle 2J+1|R\rangle=\langle 2(J+1)|R\rangle=0, \tag{38a}$$

$$\langle -2|R\rangle=\langle -3|R\rangle=\ \ldots\ =\langle -2J|R\rangle=\langle -(2J+1)|R\rangle=0. \tag{38b}$$

We then set $|j|=J+1$ in eqs (36) and (37). This means that we have to consider the two values $j=\pm(J+1)$. For the plus sign eqs (36) and (37) yield, respectively

$$\langle \tfrac{3}{2}(J+1)|R\rangle-\langle J+1|R\rangle+\langle 2(J+1)|R\rangle+\langle 2J+1|R\rangle+\langle 2J+3|R\rangle=0, \tag{39a}$$

$$\langle \tfrac{1}{2}(2J+3)|R\rangle+\langle 2J+3|R\rangle-\langle J+1|R\rangle+\langle 2(J+1)|R\rangle-\langle J+2|R\rangle+\langle 2(J+2)|R\rangle=0. \tag{39b}$$

For $j=-(J+1)$ eqs (36) and (37) give, respectively

$$\langle -\tfrac{3}{2}(J+1)|R\rangle-\langle -(J+1)|R\rangle+\langle -2(J+1)|R\rangle+\langle -(2J+3)|R\rangle+\langle -(2J+1)|R\rangle=0, \tag{40a}$$

$$\langle -\tfrac{1}{3}(2J+1)|R\rangle + \langle -(2J+1)|R\rangle - \langle -(J+1)|R\rangle + \langle -2(J+1)|R\rangle - \langle -J|R\rangle + \langle -2J|R\rangle = 0. \quad (40b)$$

In eqs (39) and (40), the coefficients $\langle J+1|R\rangle$, $\langle 2(J+1)|R\rangle$, $\langle 2J+1|R\rangle$, $\langle J+2|R\rangle$, $\langle -(J+1)|R\rangle$, $\langle -(2J+1)|R\rangle$, $\langle -J|R\rangle$, and $\langle -2J|R\rangle$, vanish by the assumptions eqs (38) and the fact that $J > 1$. The coefficients $\langle \frac{2}{3}(J+1)|R\rangle$, $\langle \frac{1}{3}(2J+3)|R\rangle$, $\langle -\frac{2}{3}(J+1|R\rangle$, and $\langle -\frac{1}{3}(2J+1)|R\rangle$ all vanish because in each case either the index is not an integer, or if an integer, the index falls in the range of eqs (38). After striking out the vanishing coefficients, the results of eqs (39) and (40) are

$$\langle 2J+3|R\rangle = \langle 2(J+1)+1|R\rangle = 0,$$

$$\langle 2(J+2)|R\rangle = \langle 2(J+1+1)|R\rangle = 0,$$

$$\langle -2(J+1)|R\rangle = 0,$$

$$\langle -(2J+3)|R\rangle = \langle -[2(J+1)+1]|R\rangle = 0.$$

These results extend the range of assumptions (38) from $|j| = J$ to $|j| = J+1$. We have therefore shown by induction that the only nonvanishing coefficients in the Fourier series eq (30) are $\langle -1|R\rangle$, $\langle 0|R\rangle$, and $\langle 1|R\rangle$.

The Fourier expansion of $R(\alpha)$ may now be written

$$R(\alpha) = \langle 0|R\rangle + \langle 1|R\rangle\, e^{i\alpha} + \langle -1|R\rangle\, e^{-i\alpha}. \quad (41)$$

The coefficients in this expansion obey the restriction (35). We now define the matrix coefficients L, M, and N from the equations

$$L \equiv \langle 0|R\rangle, \quad (42a)$$

$$M \equiv \langle 1|R\rangle + \langle -1|R\rangle, \quad (42b)$$

$$N \equiv -i(\langle 1|R\rangle - \langle -1|R\rangle). \quad (42c)$$

The restriction (35) expressed in terms of L, M, and N, is

$$L + M = I. \quad (43)$$

The expansion (41) now takes the form

$$R(\alpha) = I - M - N \sin\alpha + M\cos\alpha. \quad (44)$$

In arriving at the form (44) we have assumed that $R(\alpha)$ obeys the group property. We now have to insure that it does by imposing the group property on eq (44). For this purpose it is sufficient to require the condition $[R(\alpha)]^2 = R(2\alpha)$. This then implies eq (6) for arbitrary α_1 and α_2. Equating $R(2\alpha)$ to the square of the right side of eq (44), and expressing the quantities $\sin^2\alpha$, $\cos^2\alpha$, and $\sin\alpha\cos\alpha$ in terms of $\cos 2\alpha$ or $\sin 2\alpha$, we have

$$I - M + M\cos 2\alpha - N\sin 2\alpha = 2M(I-M)\cos\alpha - (2N - MN - NM)\sin\alpha - \tfrac{1}{2}(MN+NM)\sin 2\alpha$$

$$+\tfrac{1}{2}(M^2+N^2) + (I-M)^2 + \tfrac{1}{2}(M^2-N^2)\cos 2\alpha. \quad (45)$$

By equating the coefficients of similar Fourier components on the two sides of this equation, we arrive at the relations

$$M = -N^2, \quad (46a)$$

$$N^3 = -N. \quad (46b)$$

These relations show that all powers of N higher than second are equivalent to either N or N^2. Equation (44) may now be written

$$R(\alpha) = I + N^2 - N\sin\alpha - N^2\cos\alpha. \quad (47)$$

It remains only to determine the matrix N. This is done by imposing both the first and second orthogonality conditions on the form (47). The most convenient way to impose the first orthogonality conditions is to consider an infinitesimal rotation. This leads to the restriction

$$\tilde{N}=-N,$$

which states that N is antisymmetric. Instead of imposing the complete second orthogonality conditions it is sufficient to impose a consequence of them, namely, the existence of an axis of rotation. Therefore if n is a column matrix representing the axis of rotation, imposing the condition

$$R(\alpha)n=n, \tag{48}$$

places a restriction on N which guarantees that eq (47) obeys the second orthogonality conditions. Equation (48) becomes

$$N^2n-N^2n\cos\alpha-Nn\sin\alpha=0,$$

and the restriction which this places on N is

$$Nn=0. \tag{49}$$

If the components of n are formally regarded as known, the eqs (49) are three linear homogeneous equations for the determination of the three independent elements of N. These elements are linear functions of the components of n. The only antisymmetric matrix which can be formed linearly from n is $\epsilon_{ijk}n_j$. Hence

$$N_{ik}=a\epsilon_{ijk}n_j, \tag{50}$$

where a is a constant. It can be determined by imposing the condition (46b) on eq (50). The result is that $a^2=1$. By choosing $a=+1$, we formally complete the identification of eq (47) with the solution (5).

The exponential representation (I 28) of $R(\alpha)$ is established by noting from eq (5) (or (47)) that

$$\frac{dR}{d\alpha}=-NR. \tag{51}$$

Recognizing that $R(0)=I$, eq (51) can be formally integrated to give

$$R(\alpha)=e^{-N\alpha}. \tag{52}$$

To establish the equivalence between eqs (47) and (52), one expands eq (52) as a power series in $N\alpha$. By noting eq (46b), all powers of N which arise are expressible as either N or N^2. The resulting series is

$$R(\alpha)=I-N\left(\alpha-\frac{\alpha^3}{3!}+\frac{\alpha^5}{5!}-\ldots\right)+N^2\left(\frac{\alpha^2}{2!}-\frac{\alpha^4}{4!}+\frac{\alpha^6}{6!}-\ldots\right). \tag{53}$$

The coefficient of N can be recognized as the beginning of the series for $\sin\alpha$ while the coefficient of N^2 is seen to be the start of the series for $1-\cos\alpha$.

Note added in proof. Since the publication of papers I, II, and III, a reference [Gibbs, J. W., and Wilson, E. B., Vector Analysis, sec. 53. (Charles Scribner and Sons, New York, 1909; Dover Publications, Inc., New York, 1960)], has come to the author's attention in which the second orthogonality conditions are derived in a way which is similar to the method used in I, although not exhibited in matrix notation. Improper rotations are not taken into account. The relations which we have called the second orthogonality conditions were apparently known to mathematicians of the nineteenth century although no special name was given to them. The full exploration of the content of those equations, which has been the subject of this series of papers, does not appear to have been carried out until now.

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Minimax Error Selection of a Univariate Distribution With Prescribed Componentwise Bounds and Ranking

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The topic treated is that of finding a reproducible, plausible and computationally simple method of selecting a discrete frequency distribution with a prescribed ranking of its components and prescribed upper and lower bounds on these components. The problem is shown to be tractable when a minimax error selection criterion is employed, and "error" is measured by maximum absolute deviation among components. In this case one obtains a linear program of a special form admitting explicit solution.

Key words: Linear programs; mathematical models; minimax estimation; operations research; probability distribution.

1. Introduction

A discrete univariate probability distribution will be represented here as a probability n-vector, i.e., an n-vector $\mathbf{x}$ whose components x_i are nonnegative and sum to unity. Let P be some class of such distributions, describable as a closed subset of $\mathbf{x}$-space. Then one can pose the problem of *minimax error selection* of a member of P, i.e. of choosing $\mathbf{x}\epsilon P$ to minimize

$$F(\mathbf{x}) = \max\{d(\mathbf{x}, \mathbf{y}) : \mathbf{y}\epsilon P\} \tag{1.1}$$

where d is some metric on n-space. The particular metric

$$d(\mathbf{x}, \mathbf{y}) = \max_i |x_i - y_i| \tag{1.2}$$

will be employed, essentially because of its tractability for what follows.

In [1],[1] this selection problem was solved for the case in which the class P of admissable distributions was specified by componentwise bounds, i.e.,

$$\mathbf{O} \leqq \mathbf{L} \leqq \mathbf{x} \leqq \mathbf{U} \tag{1.3}$$

where $\mathbf{L}$ and $\mathbf{U}$ are given n-vectors. (It was also solved for a more general "weighted" version of the metric d.) In [2] it was solved for the case in which P was described by a given componentwise ranking,

$$x_1 \leqq x_2 \leqq \ldots \leqq x_n. \tag{1.4}$$

The present note gives a solution method for the case in which both types of constraint are imposed, i.e., P is specified by (1.3) and (1.4) *together*.

Some preliminaries are presented in section 2. The formulation of the solution method is then begun in section 3, and completed in section 4.

[1] Figures in brackets indicate the literature references at the end of the paper.

2. Preliminaries

This section considers three subproblems which will arise later.

PROBLEM I: Given n-vectors $\mathbf{A}$ and $\mathbf{B}$, and number S, find an n-vector $\mathbf{y}$ such that

$$\mathbf{A} \leq \mathbf{y} \leq \mathbf{B}, \tag{2.1}$$

$$\Sigma_i y_i = S, \tag{2.2}$$

$$y_1 \leq y_2 \leq \ldots \leq y_n. \tag{2.3}$$

This problem is treated in [3] (see "Problem IV" in section 2 of that paper), where a solution method is given, and the following two conditions are shown to be jointly necessary and sufficient for the existence of a solution. First,

$$A_i \leq B_j \quad \text{for} \quad i \leq j. \tag{2.4}$$

In terms of the nondecreasing sequences $\{A_i'\}_1^n$ and $\{B_i'\}_1^n$ defined by

$$A_i' = \max_{j \leq i} A_j, \qquad B_i' = \min_{j \geq i} B_j, \tag{2.5}$$

the second condition is

$$\Sigma_i A_i' \leq S \leq \Sigma_i B_i'. \tag{2.6}$$

PROBLEM II: Given n-vector $\mathbf{Z}$ and number S, find the minimum value z^* of z such that $z \geq 0$ and

$$\Sigma_i \max\{0, Z_i - z\} \leq S. \tag{2.7}$$

This problem is a special case of one treated in [3] (see "Problem II" in section 2 of that paper), where a solution method is given, and

$$S \geq 0 \tag{2.8}$$

is shown to be a necessary and sufficient condition for a solution to exist.

PROBLEM III: Given n-vector $\mathbf{Z}$ and number S, find the minimum value z^{**} of z such that $z \geq 0$ and

$$\Sigma_i \min\{Z_i, z\} \geq S. \tag{2.9}$$

This problem too is a special case of one treated in [3] (see "Problem III" in section 2 of that paper), where a solution method is given, and

$$\Sigma_i Z_i \geq S \tag{2.10}$$

is shown to be a necessary and sufficient condition for a solution to exist.

3. Analysis

We turn now to the original problem, concerning the constraint set

$$P = \{\mathbf{x}: 0 \leq \mathbf{L} \leq \mathbf{x} \leq \mathbf{U},\ \Sigma_i x_i = 1, \tag{3.1}$$

$$x_1 \leq x_2 \leq \ldots \leq x_n\}.$$

Recall that the problem is to find $\mathbf{x} \in P$ to minimize

$$F(\mathbf{x}) = \max\{\max_i |x_i - y_i| : \mathbf{y} \in P\}. \tag{3.2}$$

As a first application of the material in the preceding section, consider the question of whether

the constraints are consistent, i.e., whether P is nonempty. The existence of a $\mathbf{y} \in P$ is an instance of Problem I, with

$$\mathbf{A} = \mathbf{L}, \mathbf{B} = \mathbf{U}, S = 1.$$

Thus the necessary and sufficient conditions for consistency are that

$$L_i \leq U_j \qquad \text{for } i \leq j, \tag{3.3}$$

and, in terms of the quantities

$$L_i' = \max_{j \leq i} L_j, \quad U_i' = \min_{j \geq i} U_j, \tag{3.4}$$

that

$$\Sigma_i L_i' \leq 1 \leq \Sigma_i U_i'. \tag{3.5}$$

These conditions will be assumed satisfied from here on. Note that P can be rewritten

$$P = \{\mathbf{x} : 0 \leq \mathbf{L}' \leq \mathbf{x} \leq \mathbf{U}', \ \Sigma_i x_i = 1, \tag{3.6}$$

$$x_1 \leq x_2 \leq \ \ldots \ \leq x_n\}$$

where $\mathbf{L}'$ and $\mathbf{U}'$ are the vectors with respective components L_i' and U_i'.

Next, set

$$M_i^+ = \max \{x_i : \mathbf{x} \in P\}, \tag{3.7}$$

$$M_i^- = \min \{x_i : \mathbf{x} \in P\}; \tag{3.8}$$

the determination of these quantities will be discussed later. As in [1] and [2], we have

$$\begin{aligned} F(\mathbf{x}) &= \max_y \max_i \max \{y_i - x_i, x_i - y_i\} \\ &= \max_i \max \{\max_y (y_i - x_i), \max_y (x_i - y_i)\} \\ &= \max_i \max \{M_i^+ - x_i, x_i - M_i^-\}. \end{aligned} \tag{3.9}$$

By (3.9), the minimization of $F(\mathbf{x})$ is equivalent to the following linear program; choose number z and vector $\mathbf{x} \in P$, to minimize z subject to the further conditions

$$z \geq M_i^+ - x_i, \qquad z \geq x_i - M_i^- \qquad (\text{all } i).$$

For fixed z, the conditions on $\mathbf{x}$ read

$$\max \{L_i', M_i^+ - z\} \leq x_i \leq \min \{U_i', M_i^- + z\} \qquad (\text{all } i), \tag{3.10}$$

$$\Sigma_i x_i = 1, \tag{3.11}$$

$$x_i \leq x_2 \leq \ \ldots \ \leq x_n. \tag{3.12}$$

Thus the objective is to find the smallest z for which such an $\mathbf{x}$ exists (and to find one).

Now finding an $\mathbf{x}$ to satisfy (3.10) through (3.12) is an instance of Problem I, with $S = 1$ and

$$A_i = \max \{L_i', M_i^+ - z\}, \ B_i = \min \{U_i', M_i^- + z\}.$$

Since sequences $\{L_i'\}_1^n$ and $\{U_i'\}_1^n$ are nondecreasing, while definitions (3.7–8) and the definition of P imply that $\{M_i^+\}_1^n$ and $\{M_i^-\}_1^n$ are nondecreasing, it follows that $\{A_i\}_1^n$ and $\{B_i\}_1^n$ are nondecreasing. Hence in this case, the conditions for the existence of a solution to Problem I become

$$\max \{L_i', M_i^+ - z\} \leq \min \{U_i', M_i^- + z\} \qquad (\text{all } i), \tag{3.13}$$

$$\Sigma_i \max \{L_i', M_i^+ - z\} \leq 1, \tag{3.14}$$

$$\Sigma_i \min \{U_i', M_i^- + z\} \geq 1. \tag{3.15}$$

We seek $z_{\min}$, the smallest value of z satisfying these three conditions. The redundant condition $z \geq 0$ can also be imposed.

Let us set

z° = least z satisfying (3.13),
z^* = least nonnegative z obeying (3.14),
z^{**} = least nonnegative z obeying (3.15).

Then we have

$$z_{\min} = \max \{z^\circ, z^*, z^{**}\}; \tag{3.16}$$

this is the case since the left-hand and right-hand sides in (3.13) are respectively nonincreasing and nondecreasing functions of z, while the sums in (3.14) and (3.15) are respectively nonincreasing and nondecreasing.

Since $L_i' \leq U_i'$, the determination of z^0 involves the conditions

$$L_i' \leq M_i^- + z,\ M_i^+ - z \leq U_i',\ M_i^+ - z \leq M_i^- + z,$$

and leads to

$$z^0 = \max_i \max \{L_i' - M_i^-,\ M_i^+ - U_i',\ (M_i^+ - M_i^-)/2\}. \tag{3.17}$$

Next, (3.14) is equivalent to

$$\Sigma_i \max \{0,\ (M_i^+ - L_i') - z\} \leq 1 - \Sigma_i L_i',$$

so that determining z^* is an instance of Problem II with

$$Z_i = M_i^+ - L_i', \qquad S = 1 - \Sigma_i L_i'.$$

Finally, (3.15) is equivalent to

$$\Sigma_i \min \{U_i' - M_i^-, z\} \geq 1 - \Sigma_i M_i^-,$$

so that determining z^{**} is an instance of Problem III, with

$$Z_i = U_i' - M_i^-, \qquad S = 1 - \Sigma_i M_i^-.$$

4. Determination of M_i^+ and M_i^-

It only remains to discuss the determination of the quantities M_i^+ and M_i^- defined by (3.7) and (3.8). For this purpose, consider the conditions under which

$$\mathbf{x} = (y_1, \ldots, y_{i-1}, x_i, y_i, \ldots, y_{n-1}) \epsilon P.$$

These conditions are

$$L_i' \leq x_i \leq U_i', \tag{4.1}$$

$$L_j' \leq y_j \leq \min \{U_j', x_i\} \qquad \text{for } j < i, \tag{4.2}$$

$$\max \{L_{j+1}', x_i\} \leq y_j \leq U_{j+1}' \qquad \text{for } j \geq i, \tag{4.3}$$

$$\Sigma_j y_j = 1 - x_i, \tag{4.4}$$

$$y_1 \leqslant y_2 \leqslant \quad \ldots \leqslant y_{n-1}. \tag{4.5}$$

Now the determination of an $(n-1)$-vector $\mathbf{y}$, satisfying (4.2–5), is an instance of Problem I with $n-1$ replacing n, with $S=1-x_i$, and with

$$A_j = L'_j \text{ and } B_j = \min\{U'_j, x_i\} \qquad \text{for } j < i,$$

$$A_j = \max\{L'_{j+1}, x_i\} \text{ and } B_j = U'_{j+1} \text{ for } j \geqslant i.$$

Since $\{L'_j\}_1^n$ and $\{U'_j\}_1^n$ are nondecreasing, and (4.1) holds, the sequences $\{A_j\}_1^{n-1}$ and $\{B_j\}_1^{n-1}$ are also nondecreasing. Thus the conditions in this case, for Problem I to have a solution $\mathbf{y}$, are

$$L'_j \leqslant \min\{U'_j, x_i\} \qquad \text{for } j < i, \tag{4.6}$$

$$\max\{L'_{j+1}, x_i\} \leqslant U'_{j+1} \qquad \text{for } j \geqslant i, \tag{4.7}$$

$$\Sigma_{j<i} L'_j + \Sigma_{j>i} \max\{L'_j, x_i\} \leqslant 1 - x_i, \tag{4.8}$$

$$\Sigma_{j<i} \min\{U'_j, x_i\} + \Sigma_{j>i} U'_j \geqslant 1 - x_i. \tag{4.9}$$

Conditions (4.6) and (4.7) are automatically satisfied. Condition (4.8) can be rewritten

$$x_i + \Sigma_{j>i} \max\{L'_j, x_i\} \leqslant 1 - \Sigma_{j<i} L'_j; \tag{4.10}$$

the left-hand side is a continuous increasing function of x_i, bounded neither above nor below, which therefore equals the right-hand side for a unique value $x_i^\dagger$ of x_i. Condition (4.9) can be rewritten

$$x_i + \Sigma_{j<i} \min\{U'_j, x_i\} \geqslant 1 - \Sigma_{j>i} U'_j; \tag{4.11}$$

again the left-hand side is an increasing function of x_i, which equals the right-hand side for a unique value x_i^- of x_i. It follows that (4.8) and (4.9) are equivalent to

$$x_i^- \leqslant x_i \leqslant x_i^\dagger.$$

By combination with (4.1), this yields

$$M_i^\dagger = \min\{U'_i, x_i^\dagger\}, \tag{4.12}$$

$$M_i^- = \max\{L'_i, x_i^-\}. \tag{4.13}$$

Now the determination of $x_i^\dagger$ and x_i^- must be discussed.

To determine $x_i^\dagger$, form the quantities

$$L_k^* = (k-i)\, L'_k + \Sigma_{j \geqslant k} L'_j \qquad (k \geqslant i).$$

The sequence $\{L_k^*\}_i^n$ is nondecreasing, since

$$L_{k+1}^* - L_k^* = (k+1-i)\,(L'_{k+1} - L'_k) \geqslant 0.$$

We have

$$1 - \Sigma_{j<i} L'_j \geqslant \Sigma_{j \geqslant i} L'_j = L_i^*,$$

and so there is a last $k \in \{i, i+1, \ldots, n\}$ for which

$$1 - \Sigma_{j<i} L'_j \geqslant L_k^*.$$

For this k, it is readily verified that

$$x_i^\dagger = [1 - \Sigma_{j<i} L'_j - \Sigma_{j>k} L'_j\,]/(k+1-i) \tag{4.14}$$

satisfies $x_i^+ \epsilon [L_k', L_{k+1}']$ . . . or $x_i^+ \geq L_n'$ if $k=n$ . . . and also satisfies (4.10) as an equation.

To determine x_i^-, form the quantities

$$U_k^* = (i-k)\, U_k' + \Sigma_{j<k} U_j' \qquad (k \leq i).$$

The sequence $\{U_k^*\}_i$ is nondecreasing, since

$$U_k^* - U_{k-1}^* = (i-k+1)\,(U_k' - U_{k-1}') \geq 0.$$

We have

$$1 - \Sigma_{j>i}\ U_j' \leq \Sigma_{j<i}\ U_j' = U_i^*,$$

and so there is a *least* $k \epsilon \{1, 2, \ldots, i\}$ for which

$$1 - \Sigma_{j>i}\ U_j' \leq U_k^*.$$

For this k, it is readily verified that

$$x_i^- = \left[1 - \Sigma_{j>i}\ U_j' - \Sigma_{j<k}\ U_j'\right] / (i+1-k) \tag{4.15}$$

satisfies $x_i^- \epsilon [U_{k-1}', U_k']$ . . . or $x_i^- \leq U_1'$ if $k=1$ . . . and also satisfies (4.11) as an equation.

5. References

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Minimax Adjustment of a Univariate Distribution to Satisfy Componentwise Bounds and/or Ranking

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Consider a discrete probability distribution, represented by an n-vector $\mathbf{a}$. This paper treats the problem of adjusting $\mathbf{a}$ as little as possible, in the sense of minimizing $\max_i|x_i-a_i|$, to obtain a distribution $\mathbf{x}$ which satisfies given componentwise bounds $\mathbf{L}\leqslant\mathbf{x}\leqslant\mathbf{U}$, or a given componentwise ranking, or both. The resulting linear programs are shown to admit special explicit solution algorithms.

Key words: Linear programs; mathematical models; minimax estimation; operations research; probability distribution.

1. Introduction

An n-vector $\mathbf{x}$ will be called a *probability vector* if its components x_i are nonnegative and sum to unity. This paper deals first with the following problem: given a probability n-vector $\mathbf{a}$, and n-vectors $\mathbf{L}$ and $\mathbf{U}$, to find a probability n-vector $\mathbf{x}$ which minimizes

$$F(\mathbf{x})=\max_i|x_i-a_i| \tag{1.1}$$

subject to the constraints

$$\mathbf{O}\leqslant\mathbf{L}\leqslant\mathbf{x}\leqslant\mathbf{U}. \tag{1.2}$$

For the problem to be feasible, it is obviously necessary that

$$L_i\leqslant U_i \qquad (\text{all } i), \tag{1.3}$$

$$\Sigma_i L_i\leqslant 1\leqslant\Sigma_i U_i, \tag{1.4}$$

and so these conditions are imposed at the outset.

The motivating situation is one in which values must be attributed to the components x_i of an unknown discrete probability distribution. One type of information, e.g., data on some previous analogous situation, suggests the estimate $\mathbf{a}$. Another, e.g., theoretical analyses or subjective opinions on the "present" situation, imposes the componentwise bounds (1.2). If $\mathbf{a}$ does not satisfy (1.2), a common procedure is to "adjust $\mathbf{a}$ as little as possible" so as to satisfy (1.2). Here the minimization of $F(\mathbf{x})$ is taken to express the "as little as possible" criterion in replacing $\mathbf{a}$ by a suitable $\mathbf{x}$.

After some preliminaries are disposed of in section 2, a solution method for this problem is presented in section 3. In fact, the method is developed for the more general version in which (1.1) is replaced by

$$F(\mathbf{x})=\max_i\{w_i|x_i=a_i|\}, \tag{1.5}$$

where the w_i are prescribed positive "weights." This corresponds to the case in which the accuracy, with which $\mathbf{x}$ approximates $\mathbf{a}$, is more important for some components than for others.

Section 4 considers the analogous problem in which the componentwise-bound constraints (1.2) are replaced by a given componentwise ranking

$$x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n. \tag{1.6}$$

Then, in section 5, the problem with *both* bounds *and* ranking is treated. (Both of these sections deal only with the "unweighted" objective function (1.1); the weighted version can be handled as a linear program, but our concern is with more explicit methods.)

Related work is found in [1, 2, 3][1]. The present paper, though self-contained, isolates in a form more convenient for reference some material appearing in [1].

2. Preliminaries

This section contains solution methods for four subproblems which will arise later. The material is presented for the sake of completeness; the same or similar problems have surely arisen in the literature as phases in other optimization analyses. In each case below, the solution method provides a constructive proof that certain obviously necessary conditions, for the existence of solutions, are also sufficient.

PROBLEM I. Given n-vectors $\mathbf{A}$ and $\mathbf{B}$, and number S, find an n-vector $\mathbf{y}$ such that

$$\mathbf{A} \leqslant \mathbf{y} \leqslant \mathbf{B}, \tag{2.1}$$

$$\Sigma_i y_i = S. \tag{2.2}$$

The conditions

$$\mathbf{A} \leqslant \mathbf{B}, \tag{2.3}$$

$$\Sigma_i A_i \leqslant S \leqslant \Sigma_i B_i, \tag{2.4}$$

which are clearly necessary for Problem I to have a solution, will be assumed to hold. If $S = \Sigma_i B_i$ then $\mathbf{y} = \mathbf{B}$ is clearly a solution, so we assume $S < \Sigma_i B_i$. Then $k = n$ has the property

$$\Sigma_{j \leqslant k} B_j + \Sigma_{j > k} A_j > S, \tag{2.5}$$

but $k = 0$ does not, and so there is a smallest $k \in \{1, 2, \ldots, n\}$ with this property. For that k, not only (2.5) but also

$$\Sigma_{j < k} B_j + \Sigma_{j \geqslant k} A_j \leqslant S \tag{2.6}$$

holds. Now set

$$y_j = B_j \qquad \text{for } j < k,$$

$$y_j = A_j \qquad \text{for } j > k,$$

$$y_k = S - \dot{\Sigma}_{j \neq k} y_j;$$

that $A_k \leqslant y_k \leqslant B_k$ follows from (2.5) and (2.6).

PROBLEM II. Given n-vector $\mathbf{Z}$, positive n-vector $\mathbf{w}$, and number S, find the minimum value z^* of z such that $z \geqslant 0$ and

$$\Sigma_1^n \max \{0, Z_i - z/w_i\} \leqslant S. \tag{2.7}$$

The condition $S \geqslant 0$, which is clearly necessary for Problem II to have a solution, will be assumed. Moreover, if $S = 0$, then each of the nonnegative summands in (2.7) would have to vanish, yielding

$$z^* = \max \{0, \max_i w_i Z_i\}$$

as the solution; thus from now on we assume $S > 0$. Furthermore if $Z_i \leqslant 0$, then the ith summand in (2.7) will vanish for *every* $z \geqslant 0$, and so such Z_i can be deleted from the problem in advance; if none are left then clearly $z^* = 0$ is the solution. So $Z_i > 0$ will be assumed.

[1] Figures in brackets indicate the literature references at the end of the paper.

Now choose $Z_{n+1}=0$ and any $w_{n+1}>0$, renumber so that

$$w_1Z_1 \geq w_2Z_2 \geq \cdots \geq w_nZ_n > w_{n+1}Z_{n+1}=0,$$

and set

$$Z_j^* = \Sigma_1^j Z_i - w_jZ_j \Sigma_1^j (1/w_i).$$

The sequence $\{Z_j^*\}_1^{n+1}$ is given by the recursion

$$Z_{j+1}^* = Z_j^* + (w_jZ_j - w_{j+1}Z_{j+1})\ \Sigma_1^j (1/w_i),$$

which shows it to be nondecreasing. And *unless* $\Sigma_1^n Z_i < S$ (in which case $z^*=0$ is the solution), we have

$$Z_1^* = 0 < S \leq \Sigma_1^n Z_i = Z_{n+1}^*.$$

Thus there is a unique $J \epsilon \{1, 2, \ldots, n\}$ such that

$$0 = Z_1^* \leq Z_2^* \leq \ldots \leq Z_J^* < S \leq Z_{J+1}^* \leq \ldots \leq Z_{n+1}^*. \tag{2.8}$$

If $0 \leq z < w_{J+1}Z_{J+1}$, then

$$\Sigma_i \max\{0, Z_i - z/w_i\} \geq \Sigma_1^{J+1}\ Z_j - z\ \Sigma_1^{J+1} (1/w_j) > Z_{J+1}^* \geq S,$$

so that z does not satisfy (2.7). But if $w_{J+1}Z_{J+1} \leq z \leq w_JZ_J$, then (2.7) becomes

$$\Sigma_1^j Z_i - z\ \Sigma_1^j (1/w_i) \leq S$$

which is equivalent to

$$z \geq z^* = (\ \Sigma_1^j Z_j - S)/\Sigma_1^j (1/w_i). \tag{2.9}$$

By use of (2.8), the value of z^* proposed in (2.9) is easily verified to satisfy $w_{J+1}Z_{J+1} \leq z \leq w_JZ_J$, and so is indeed the smallest $z \geq 0$ obeying (2.7).

PROBLEM III. Given n-vector $\mathbf{Z}$, positive n-vector $\mathbf{w}$ and number S, find the minimum value z^{**} of z such that $z \geq 0$ and

$$\Sigma_1^n \min\{Z_i,\ z/w_i\} \geq S. \tag{2.10}$$

The condition

$$\Sigma_1^n Z_i \geq S, \tag{2.11}$$

which is obviously necessary if (2.10) is to have a solution, will be assumed. If equality holds in (2.11), then for each i the ith summand in (2.10) must equal Z_i, so that the solution is

$$z^{**} = \max\{0, \max_i w_iZ_i\};$$

thus from now on we assume strict inequality in (2.11). Moreover, if $Z_i < 0$ then for any $z \geq 0$, Z_i could be replaced by 0 on the left-hand side of (2.10) without change in value; hence it can be assumed that all $Z_i \geq 0$. Now $z^{**}=0$ will give the solution if $S \leq 0$, so we also assume $S > 0$.

Choose $Z_{n+1}=0$ and any $w_{n+1}>0$, renumber so that

$$w_1Z_1 \geq w_2Z_2 \geq \ldots \geq w_nZ_n \geq w_{n+1}Z_{n+1}=0,$$

and set

$$Z_j^{**} = w_jZ_j \Sigma_1^j (1/w_i) + \Sigma_{j+1}^n Z_i.$$

The sequence $\{Z_j^{**}\}_1^{n+1}$ obeys the recursion

$$Z_{j+1}^{**} = Z_j^{**} + (w_{j+1}Z_{j+1} - w_jZ_j)\ \Sigma_1^j\ (1/w_i)$$

and so is nonincreasing. Since

$$Z_{n+1}^{**} = 0 < S < \Sigma_1^n Z_i = Z_1^{**},$$

there is a unique $J \epsilon \{1, 2, \ldots, n\}$ such that

$$Z_1^* \geq Z_2^* \geq \ldots \geq Z_J^* \ \geq S > Z_{J+1}^* \geq \ldots \geq Z_{n+1}^* = 0. \tag{2.12}$$

Now if $z \leq w_{J+1}$, then

$$\Sigma_i \min\{Z_i, z/w_i\} \leq z\Sigma_1^{J+1}(1/w_i) + \Sigma_{J+2}^n Z_1 \leq Z_{J+1}^{**} < S$$

so that z does not satisfy (2.10). But if $w_{J+1}Z_{J+1} \leq z \leq w_JZ_J$, then (2.10) becomes

$$z\Sigma_1^J(1/w_i) + \Sigma_{J+1}^n Z_j \leq S$$

which is equivalent to

$$z \geq z^{**} = (S - \Sigma_{J+1}^n Z_j)/\Sigma_1^J(1/w_i). \tag{2.13}$$

By use of (2.12), the value of z^{**} proposed in (2.13) is easily verified to satisfy $w_{J+1}Z_{J+1} \leq z \leq w_JZ_J$, and so is indeed the smallest $z \geq 0$ obeying (2.10).

PROBLEM IV. Given n-vectors $\mathbf{A}$ and $\mathbf{B}$, and number S, find an n-vector $\mathbf{y}$ such that

$$\mathbf{A} \leq \mathbf{y} \leq \mathbf{B}, \tag{2.14}$$

$$\Sigma_i y_i = S, \tag{2.15}$$

$$y_1 \leq y_2 \leq \ldots \leq y_n. \tag{2.16}$$

Here it is convenient to define nondecreasing sequences $\{A_i'\}_1^n$ and $\{B_i'\}_1^n$, forming the components of respective vectors $\mathbf{A}'$ and $\mathbf{B}'$, by

$$A_i' = \max_{j \leq i} A_j, \qquad B_i' = \min_{j \geq i} B_j. \tag{2.17}$$

Then (2.16) and (2.14) are readily proved equivalent to (2.16) and

$$\mathbf{A}' \leq \mathbf{y} \leq \mathbf{B}'. \tag{2.14'}$$

Thus necessary conditions, for Problem IV to have a solution, are

$$\mathbf{A}' \leq \mathbf{B}' \qquad (\text{i.e., } A_i \leq B_j \text{ for } i \leq j), \tag{2.18}$$

$$\Sigma_i A_i' \leq S \leq \Sigma_i B_i' \quad . \tag{2.19}$$

These will be assumed to hold.

If $\mathbf{A}' = \mathbf{B}'$, then $\mathbf{y} = \mathbf{A}' = \mathbf{B}'$ is the solution. For $\mathbf{A}' \neq \mathbf{B}'$, define

$$\theta = [S - \Sigma_i A_i']/[\Sigma_i B_i' - \Sigma_i A_i']$$

and set

$$\mathbf{y} = \mathbf{A}' + \theta(\mathbf{B}' - \mathbf{A}').$$

Then (2.15) follows from the choice of θ, and (2.14′) becomes $0 \leq \theta \leq 1$, which follows from (2.19). As for (2.16), $i \leq j$ implies that $A_i' \leq A_j'$ and $B_i' \leq B_j'$, so that

$$y_i = (1-\theta)A_i' + \theta B_i' \leq (1-\theta)A_j' + \theta B_j' = y_j .$$

(The same approach yields a simpler solution method for Problem I than the one given above.)

3. Solution for Componentwise Bounds

We now return to the problem posed at the beginning of the paper, with objective function (1.5). It can be rephrased as the following linear program: choose a number z and a probability n-vector $\mathbf{x}$, to minimize z subject to the conditions

$$\mathbf{0} \leq \mathbf{L} \leq \mathbf{x} \leq \mathbf{U}, \tag{3.1}$$

$$z \geq w_i(x_i - a_i) \qquad (\text{all } i), \tag{3.2}$$

$$z \geq w_i(a_i - x_i) \qquad (\text{all } i). \tag{3.3}$$

The constraints of the linear program, including the requirement that $\mathbf{x}$ be a probability vector, can be written as follows:

$$\max\{L_i, a_i - z/w_i\} \leq x_i \leq \min\{U_i, a_i + z/w_i\} \qquad (\text{all } i), \tag{3.3a}$$

$$\Sigma_i x_i = 1. \tag{3.3b}$$

A redundant constraint $z \geq 0$ can also be imposed. Thus the aim is to determine the smallest $z \geq 0$ for which the system (3.3a), (3.3b) has a solution $\mathbf{x}$.

For any fixed z, the system is an instance of Problem I in Section 2, with $S = 1$ and

$$A_i = \max\{L_i, a_i - z/w_i\}, \qquad B_i = \min\{U_i, a_i + z/w_i\}.$$

By the analysis in Section 2, a solution $\mathbf{x}$ exists if and only if

$$\max\{L_i, a_i - z/w_i\} \leq \min\{U_i, a_i + z/w_i\} \qquad (\text{all } i), \tag{3.4}$$

$$\Sigma_i \max\{L_i, a_i - z/w_i\} \leq 1, \tag{3.5}$$

$$\Sigma_i \min\{U_i, a_i + z/w_i\} \geq 1. \tag{3.6}$$

So the objective is to determine the smallest value $z_{\min}$ of z which will satisfy (3.4), (3.5) and (3.6).

Now the left-hand side in (3.4) is nonincreasing in z, while the right-hand side is nondecreasing. The left-hand sides of (3.5) and (3.6) are respectively nonincreasing and nondecreasing in z. It follows that, if

$$z^\circ = \text{least } z \text{ obeying (3.4)},$$

$$z^* = \text{least nonnegative } z \text{ obeying (3.5)},$$

$$z^{**} = \text{least nonnegative } z \text{ obeying (3.6)},$$

then

$$z_{\min} = \max\{z^\circ, z^*, z^{**}\}. \tag{3.7}$$

Since $L_i \leq U_i$, (3.4) reduces to

$$a_i - z/w_i \leq U_i, \; L_i \leq a_i + z/w_i \qquad (\text{all } i),$$

and so z° is readily determined as

$$z^\circ = \max\{\max_i w_i(a_i - U_i), \max_i w_i(L_i - a_i)\}. \tag{3.8}$$

Next, (3.5) can be rewritten

$$\Sigma_i \max\{0, (a_i - L_i) - z/w_i\} \leq 1 - \Sigma_i L_i, \tag{3.9}$$

so that determining z^* is an instance of Problem II in Section 2, with

$$\mathbf{Z} = \mathbf{a} - \mathbf{L}, \qquad S = 1 - \Sigma_i L_i .$$

The feasibility condition $S \geq 0$ is satisfied by virtue of the first part of (1.4).

Finally, (3.6) is equivalent to

$$\Sigma_i \min\{U_i - a_i, z/w_i\} \geq 0, \tag{3.10}$$

so that the determination of z^{**} is an instance of Problem III in Section 2, with

$$\mathbf{Z} = \mathbf{U} - \mathbf{a}, \qquad S = 0.$$

The feasibility condition $\Sigma_i^n Z_i \geq S$ is satisfied by virtue of the second part of (1.4).

With z°, z^*, and z^{**} determined, $z_{\min}$ can be found from (3.7). Then a single optimizing $\mathbf{x}$ can be found by applying, to the previously-mentioned instance of Problem I with $z = z_{\min}$, the solution method given in Section 2. Concerning the nonuniqueness of $\mathbf{x}$, compare Section 5 of [1].

4. Solution for Componentwise Ranking

The next problem to be considered is the determination of a probability n-vector $\mathbf{x}$, among those which obey the componentwise ranking

$$x_1 \leq x_2 \leq \ldots \leq x_n, \tag{4.1}$$

which minimizes

$$F(\mathbf{x}) = \max_i |x_i - a_i|.$$

This too can be reformulated as a linear program, namely to select a number z and a vector $\mathbf{x} \geq 0$, so as to minimize z subject to

$$\Sigma_i x_i = 1, \tag{4.2}$$

$$z \geq x_i - a_i \qquad (\text{all } i), \tag{4.3}$$

$$z \geq a_i - x_i \qquad (\text{all } i). \tag{4.4}$$

Conditions (4.3) and (4.4), together with $\mathbf{x} \geq 0$, can be abbreviated to

$$\max\{0, a_i - z\} \leq x_i \leq a_i + z. \tag{4.5}$$

A redundant constraint $z \geq 0$ can also be imposed. Thus the aim is to determine the smallest $z \geq 0$ for which the system (4.1), (4.2), (4.5) has a solution $\mathbf{x}$.

For any fixed $z \geq 0$, the system is an instance of Problem IV in Section 2, with $S = 1$ and

$$A_i = \max\{0, a_i - z\}, B_i = a_i + z.$$

It is convenient to define vectors $\mathbf{a}^*$ and $\mathbf{a}^{**}$, with nondecreasing component sequences given by

$$a_i^* = \max_{j \leq i} a_i, \; a_i^{**} = \min_{j \geq i} a_i.$$

Then the vectors $\mathbf{A}'$ and $\mathbf{B}'$, described in Section 2's analysis of Problem 4, are given by

$$A_i' = \max\{0, a_i^* - z\}, B_i' = a_i^{**} + z.$$

The conditions (2.18) and (2.19), for Problem IV to be feasible, become

$$\max\{0, a_i - z\} \leq a_j + z \text{ for } i \leq j, \tag{4.6}$$

$$\Sigma_i \max\{0, a_i^* - z\} \leq 1 \leq \Sigma_i (a_i^{**} + z). \tag{4.7}$$

Now the objective is to find $z_{\min}$, the smallest $z \geq 0$ satisfying (4.6) and (4.7). Arguing as in Section 3, one finds that if

$z^0 =$ least nonnegative z obeying (4.6),

$z^* =$ least nonnegative z obeying first part of (4.7),

$z^{**} =$ least nonnegative z obeying second part of (4.7),

then

$$z_{\min} = \max\{z^0, z^*, z^{**}\}. \tag{4.8}$$

Since $z \geq 0$ and each $a_j \geq 0$, z^0 is readily determined from (4.6) as

$$z^0 = \max\{0, \max_{i \leq j} (a_i - a_j)/2\}. \tag{4.9}$$

Since $a_i^{**} \leq a_i$, implying

$$\Sigma_i a_i^{**} \leq \Sigma_i a_i = 1,$$

z^{**} is readily determined from (4.7) as

$$z^{**} = (1 - \Sigma_i a_i^{**})/n. \tag{4.10}$$

Finally, the determination of z^* is an instance of Problem II, with $S = 1$, $w_i = 1$, and $Z_i = a_i^*$.

5. Solution for Componentwise Bounds and Ranking

The final version to be treated is the determination of a probability n-vector $\mathbf{x}$ which minimizes $\max_i |x_i - a_i|$, subject *both* to the ranking condition

$$x_1 \leq x_2 \leq \ldots \leq x_n,$$

and to the componentwise bounds

$$\mathbf{O} \leq \mathbf{L} \leq \mathbf{x} \leq \mathbf{U}.$$

The analysis proceeds much as in the preceding Section, with (4.5) replaced by

$$\max\{L_i, a_i - z\} \leq x_i \leq \min\{U_i, a_i + z\}. \tag{5.1}$$

Again we have an instance of Problem IV, with $S = 1$, and with

$$A_i = \max\{L_i, a_i - z\}, \qquad B_i = \min\{U_i, a_i + z\}.$$

As before, set

$$a_i^* = \max_{j \leq i} a_j, \; a_i^{**} = \min_{j \geq i} a_j,$$

and also put

$$L_i' = \max_{j \leq i} L_j, \qquad U_i' = \min_{j \geq i} U_j.$$

Then the vectors $\mathbf{A}'$ and $\mathbf{B}'$ are now given by

$$A_i' = \max\{L_i', a_i^* - z\}, \qquad B_i' = \min\{U_i', a_i^{**} + z\}. \tag{5.2}$$

We continue to have

$$z_{\min} = \max\{z^0, z^*, z^{**}\}, \tag{5.3}$$

where

$$z^0 = \text{least } z \text{ for which } \mathbf{A}' \leqslant \mathbf{B}',$$

$$z^* = \text{least nonnegative } z \text{ for which } \Sigma_i A_i' \leqslant 1,$$

$$z^{**} = \text{least nonnegative } z \text{ for which } \Sigma_i B_i' \geqslant 1.$$

The problem defining z^0, and hence the overall problem, has a solution only if

$$L_i \leqslant U_j \qquad \text{for } i \leqslant j. \tag{5.4}$$

Assuming this holds, the remaining conditions defining z^0 are (for $i \leqslant j$)

$$L_i \leqslant a_j + z,\ a_i - z \leqslant U_j,\ a_i - z \leqslant a_j + z,$$

and so

$$z^0 = \max_{i \leqslant j} \max\{L_i - a_j,\ a_i - U_j,\ (a_i - a_j)/2\}. \tag{5.5}$$

The condition defining z^* reads

$$\Sigma_i \max\{L_i', a_i^* - z\} \leqslant 1,$$

which can be rewritten

$$\Sigma_i \max\{0,\ (a_i^* - L_i') - z\} \leqslant 1 - \Sigma_i L_i'.$$

This is an instance of Problem II, with $w_i = 1$,

$$Z_i = a_i^* - L_i', \qquad S = 1 - \Sigma_i L_i'.$$

The consistency condition $S \geqslant 0$, i.e.,

$$\Sigma_i L_i' \leqslant 1, \tag{5.6}$$

is required for a solution to exist.

The condition defining z^{**} reads

$$\Sigma_i \min\{U_i', a_i^{**} + z\} \geqslant 1,$$

which can be rewritten

$$\Sigma_i \min\{U_i' - a_i^{**}, z\} \geqslant 1 - \Sigma_i a_i^{**}.$$

This is an instance of Problem III, with $w_i = 1$,

$$Z_i = U_i' - a_i^{**}, \qquad S = 1 - \Sigma_i a_i^{**}.$$

The consistency condition $\Sigma_i Z_i \geqslant S$, i.e.,

$$\Sigma_i U_i' \geqslant 1 \tag{5.7}$$

is required for a solution to exist.

Thus the conditions on the data $\mathbf{L}$ and $\mathbf{U}$, for the componentwise bounds and ranking to be consistent, are given by (5.4), (5.6) and (5.7).

6. References

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On the Application of Some Interpolating Functions in Physics*

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J. M. Whittaker considered interpolating equidistant samples by using the cardinal series. In this paper we consider the extension provided by the generalized sampling theorem of Kramer for nonequidistant samples. We calculate some sampling functions for cases of interest in mathematical physics. In particular, this includes the S-matrix contribution due to Regge poles, especially when a series expansion other than that of Fourier-Legendre type is needed.

Key words: Cardinal series; interpolation; Regge poles; special functions.

1. Introduction

J. M. Whittaker [1][1] considered the interpolation of functions sampled at equidistant points. This resulted in his cardinal series and its integral representation for the functions as finite Fourier transforms. The result was introduced to Communications Theory by Shannon [2] in the form: if

$$f(t)=\int_{-a}^{a} e^{ixt}g(x)dx, \tag{1}$$

where

$$g(x)\epsilon L^2(-a, a), \text{ then}$$

$$f(t)=\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{a}\right)\frac{\sin\,(at-n\pi)}{at-n\pi}. \tag{2}$$

J. M. Whittaker came verv close to touching the question of interpolating functions sampled at nonequidistant points when he considered a more general partial fraction series with samples at $\{C_n\}$, a strictly increasing sequence of positive numbers which is square summable. He noted that his cardinal series and its finite Fourier transform representation did not apply but hinted that a similar theorem would hold if $C_n=j_{0,n}$, the nth zero of the Bessel function $J_0(x)$. So it is no surprise to find Bessel functions among the first examples of the following Kramer sampling theorem [3].

Let I be an interval. Suppose that for each real t

$$f(t)=\int_I K(t, x)g(x)dx, \tag{3}$$

where $g(x)\,\epsilon\, L^2(I)$. Suppose also that for each real t, $K(t, x)\,\epsilon\, L^2(I)$ and that there exists a countable set $E=\{t_n\}$ such that $\{K(t_n, x)\}$ is a complete orthogonal set on $L^2(I)$. Then

$$f(t)=\lim_{N\to\infty}\sum_{n|<N} f(t_n)S_n(t), \tag{4}$$

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where

$$S_n(t)=\frac{\int_I K(t,x)\overline{K(t_n,x)}\,dx}{\int_I |K(t_n,x)|^2 dx} \tag{5}$$

Kramer showed that these conditions on the kernel $K(t, x)$ are satisfied by the solution of any nth order, self-adjoint differential equation and gave as illustrations a first-order equation and the Bessel equation. Before Kramer, Weiss [4] had arrived at the same result for $n=2$. Kramer illustrated this theorem for the case of $K(t, x)=J_m(xt)$ with the sampling function

$$S_n(t)=\frac{2j_{m,n}J_m(t)}{(j_{m,n}^2-t^2)J_{m+1}(j_{m,n})}, \qquad J_m(j_{m,n})=0 \tag{6}$$

and Campbell [5] illustrated it for the case of $K(t, x)=P_{t-1/2}(x)$, the Legendre function, to obtain

$$S_n(t)=\frac{2n+1}{\pi}\,\frac{\sin(t-n-\frac{1}{2})}{(t-n-\frac{1}{2})(t+n+\frac{1}{2})} \tag{7}$$

In the next section we calculate $S_n(t)$ for the case of the associated Legendre functions, the Gegenbauer functions, a Chebyshev function, the prolate spheroidal functions and a Bessel-like function satisfying a fourth-order differential equation. For more details of the calculations one may consult [6].

2. Further Application of the Kramer Generalized Sampling Theorem

(i) *Associated Legendre functions.* Consider the Legendre equation [7, p. 121]:

$$(1-z^2)\frac{d^2u}{dz^2}-2z\frac{du}{dz}+\left[t(t+1)-\frac{\mu^2}{1-z^2}\right]u=0 \tag{8}$$

where z, t, μ are unrestricted. The solution is given by:

$$u(z)=P_t^{\mu}(z)=\frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\mu/2}{}_2F_1\left(-t,t+1;1-\mu;\ \frac{1}{2}-\frac{z}{2}\right),\ |1-z|<2. \tag{9}$$

To obtain the sampling function $S_n(t)$ in (5) we need the integral $\int_{-1}^{1} P_t^m(x)P_n^m(x)\,dx$, where n and m are integers and t is unrestricted. We substitute the infinite series expansion for $P_t^m(x)$ [7, p. 167] and integrate term by term to obtain

$$S_n(t)=(-1)^m\,\frac{2n+1}{\pi}\,\frac{(n-m)!\,\Gamma(t+m+1)}{(n+m)!\,\Gamma(t-m+1)}\,\frac{\sin\pi(t-n)}{(t-n)(t+n+1)} \tag{10}$$

(ii) *Gegenbauer functions:* Consider the differential equation [7, p. 178]:

$$\frac{d}{dz}\left[(1-z^2)(1-z^2)^{\nu-1/2}\frac{du}{dz}\right]+\lambda_t(1-z^2)^{\nu-1/2}u=0,\ \lambda_t=t(t+2\nu), \tag{11}$$

satisfied by the Gegenbauer functions $C_t^{\nu}(x)$. For $S_n(t)$ in (5) we need $\int_{-1}^{1} C_t^{\nu}(x)C_n^{\nu}(x)(1-x^2)^{\nu-1/2}dx$, t being an unrestricted real parameter. We use [7, p. 159, eq (27), p. 178, and eq (23), p. 140] to obtain

$$C_t^{m+1/2}(x)=\frac{2^m m!(z^2-1)^{-\frac{m}{2}}}{(2m)!}P_{t+m}^m(x),\ m+\frac{1}{2}>0. \tag{12}$$

If we combine this result with (10) we obtain

$$S_n(t)=\frac{2}{\pi}\left[\frac{2^{2m}m!}{(2m)!}\right]^2\frac{n!(n+m+\frac{1}{2})\Gamma^2(m+\frac{1}{2})\Gamma(t+2m+1)\sin\pi(t-n)}{(2m+n)!(t+n+2m+1)\Gamma(t+1)\pi(t-n)}, \tag{13}$$

$$m+\frac{1}{2}>0.$$

(iii) *Chebyshev functions.* These are defined as $T_t(x)=C_t^0(x)$ and $U_t(x)=C_t^1(x)$ but we cannot use (13) for either. To treat the latter case we write

$$C_t^\mu(x)=\frac{2^{\mu-1/2}\Gamma(t+2\mu)\Gamma(\mu+\frac{1}{2})}{\Gamma(2\mu)\Gamma(t+1)}(1-x^2)^{\frac{1-2\mu}{4}}P_{t+\mu-1/2}^{-1/2-\mu}(x), \qquad \operatorname{Re}\mu>0; \tag{14}$$

compare (12), and use [7 p. 169, eq (1)]

$$\int_{-1}^{1}P_t^\mu(x)P_\sigma^\mu(x)dx=\frac{1}{(t-\sigma)(t+\sigma+1)}[x(t-\sigma)P_t^\mu(x)P_\sigma^\mu(x)+(\sigma+\mu)P_t^\mu(x)P_{\sigma-1}^\mu(x) \tag{15}$$

$$-(t+\mu)P_{t-1}^\mu(x)P_\sigma^\mu(x)]_{-1}^1$$

to obtain

$$\int_{-1}^{1}U_t(x)U_n(x)(1-x^2)^{1/2}dx=\int_{-1}^{1}C_t^1(x)C_n^1(x)(1-x^2)^{1/2}dx$$

$$=\frac{1}{(t-n)(t+n+2)}\cdot\frac{2\Gamma(t+2)\Gamma(n+2)\left[\Gamma\left(\frac{3}{2}\right)\right]^2}{[\Gamma(2)]^2\Gamma(t+1)n!}[x(t-n)P_{t+1/2}^{-1/2}(x)P_{n+1/2}^{-1/2}(x)$$

$$+nP_{t+1/2}^{-1/2}(x)P_{n-1/2}^{-1/2}(x)-tP_{t-1/2}^{-1/2}(x)P_{n+1/2}^{-1/2}(x)]_{-1}^1. \tag{16}$$

This result plus the orthogonal property

$$\int_{-1}^{1}\{U_n(x)\}^2(1-x^2)^{1/2}dx=\pi$$

yields the required $S_n(t)$.

The result for $T_t(x)=C_t^0(x)$, the Chebyshev function of the first kind is not available, since the above treatment is limited by the condition $\operatorname{Re}\mu>0$ in (14).

(iv) *Spheroidal wave functions.* Consider the differential equation [8, p. 134, eq (1)]:

$$(1-z^2)\frac{d^2y}{dz^2}-2z\frac{dy}{dz}+\left[\lambda+4\theta(1-z^2)-\frac{\mu^2}{1-z^2}\right]y=0, \tag{17}$$

where λ, θ, μ are given real or complex parameters, z is a complex variable and the eigenvalue $\lambda_t^\mu(\theta)=t(t+1)$ at $\theta=0$. For $\mu=m$, an integer, a solution of (17) which remains bounded at $z=1$ is the prolate spheroidal function, $Ps_t^m(x,\theta)$. For calculating $S_{n'}(t)$ in (5) we need the value of the integral $\int_{-1}^{1}Ps_t^m(x,\theta)Ps_{n'}^m(x,\theta)dx$. First we substitute [8, p. 138, eq (22)],

$$Ps_t^m(x,\theta)=\sum_{r=-\infty}^{\infty}(-1)^r a_{t,r}^m(\theta)P_{t+2r}^m(x), \tag{18}$$

then integrate term by term, using the infinite series expansion for $P^m_{l+2r}(x)$ in terms of $P^m_n(x)$ [7, p. 167] and the result [8, p. 158, eq (1)],

$$\int_{-1}^{1} Ps^m_n(x)P^m_n(x)dx = \begin{cases} (-1)^r a^m_{n',r}(\theta)\,\dfrac{(n+m)!}{(n+\frac{1}{2})(n-m)!}, & n-n'=2r, \\ \\ 0, & n-n' \text{ negative or odd}, \end{cases} \tag{19}$$

to obtain

$$\int_{-1}^{1} Ps^m_{n'}(x,\theta)Ps^m(x,\theta)dx = 2\sum_{r=0}^{\infty} a^m_{n',r}\,a^m_{l,r}\,\frac{\Gamma(t+2r+m+1)\sin\pi(t-n')}{\Gamma(t+2r-m+1)(t+n'+4r+1)\pi(t-n')} \tag{20}$$

We note that this new result reduces to the special case quoted in [8, p. 147, eq (6)], since if we let $t=n=n'$ in (20) we arrive at

$$\int_{-1}^{1} Ps^m_{n'}(x,\theta)Ps^m_{n'}(x,\theta)dx = \delta_{nn'}\sum_{r=0}^{\infty}(a^m_{n',r})^2\,\frac{(n'+2r+m)!}{(n'+2r-m)!}\cdot\frac{1}{n'+2r+\frac{1}{2}},$$

$$2r \geq m-n', \tag{21}$$

and when [8, p. 147, eq (7)] is used this becomes

$$\int_{-1}^{1} [Ps^m_n(x)]^2dx = \frac{1}{n+\frac{1}{2}}\,\frac{(n+m)!}{(n-m)!}. \tag{22}$$

To evaluate $S_n(t)$ in (5) we merely substitute (20) and (22).

(v) *Fourth-Order Differential Equations.* The following self-adjoint boundary value problem is of interest in signal detection [9]

$$[D^4-(m^2+\bar{m}^2)D^2+m^2\bar{m}^2]y(te^{-x}) = t^2e^{-2x}y(te^{-x}).$$

$$y(t)=0,\ y'(t_n)=0,\ n=1,2,\ldots. \tag{23}$$

If we take a new variable $u=e^{-x}$, the solution $H_m(t_nu)$ is a Bessel-like function which satisfies Kramer's conditions. To obtain $S_n(t)$ for (5) we need the integral $\int_0 uH_m(tu)H_m(t_nu)du$. For this we consider (23) with t replaced by t_n, and eliminate the constant coefficient terms between this equation and the original (23) to arrive at

$$S_n(t) = \frac{2tH'_m(t)}{(t^2-t_n^2)H''_m(t_n)}.$$

3. Applications in Physics

The scattering amplitude of two spinless particles interacting via a spherically symmetric potential may be written in the form [10]

$$A(k,z) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)[S_l(k)-1]P_l(z), \tag{25}$$

where k is the wave number, $z=\cos\theta$, $P_l(x)$ is the Legendre polynomial, S_l is the lth eigenvalue of the unitary S-matrix, and

$$S_l-1 = ik\int_{-1}^{1} A(k,z)P_l(z)dz. \tag{26}$$

When the $A(k, z)$ of (25) is represented by the Watson transform as a contour integral in the λ-plane, where λ represents the complex angular momentum, it is found that the contribution to $A(k, z)$ from the nth Regge pole of the S-matrix at $\lambda = \alpha_n$ with residue β_n is

$$\frac{i\pi}{k}(\alpha_n + \tfrac{1}{2})\beta_n P_{\alpha_n}(-z)/\sin(\pi\alpha_n). \tag{27}$$

To obtain the $S_l^{(n)} - 1$ corresponding to this term, eq (26) may be used. Hence we need to evaluate the integral $\int_{-1}^{1} P_{\alpha_n}(-z)P_l(z)\,dz$, which is the special case of (5) that corresponds to the Legendre function $P_{\alpha_n}(x) = K(\alpha_n, x)$. This integral is given in (7) and was used in [10] to give

$$S_l^{(n)} - 1 = \frac{\beta_n(2\alpha_n + 1)}{(l - \alpha_n)(l + \alpha_n + 1)}. \tag{28}$$

We point out that at times it is necessary to obtain series expansions for $A(k, z)$ in terms of other orthogonal functions such as the associated Legendre polynomials, the Gegenbauer functions, the Chebyshev polynomials, the prolate spheroidal functions and the Bessel-like functions. In such cases a Watson-like transform corresponding to each of these orthogonal functions may be defined in a similar way. Furthermore, for each of these transforms, the $S_l^{(n)} - 1$ term corresponding to the nth Regge pole contribution to $A(k, z)$ may be obtained in a way similar to that used for (28). This calculation may be carried out by using the sampling functions corresponding to the orthogonal functions in the kernel of the transforms. These were derived in (10), (13), (16), (20), and (24).

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Commutator Groups and Algebras*

L. Greenberg**

(March 12, 1969)

Let H and K be connected, Lie subgroups of a Lie group G. The group $[H, K]$, generated by all commutators $hkh^{-1}k^{-1}(h\epsilon H, k\epsilon K)$ is arcwise connected. Therefore, by a theorem of Yamabe, $[H, K]$ is a Lie subgroup. If $\mathfrak{H}$, $\mathfrak{K}$ denote the Lie algebras of H and K, respectively, then the Lie algebra of $[H, K]$ is the smallest algebra containing $[\mathfrak{H}, \mathfrak{K}]$, which is invariant under $ad\mathfrak{H}$ and $ad\mathfrak{K}$. An immediate consequence is that if H and K are complex Lie subgroups, then $[H, K]$ is also complex.

Key words: Adjoint representation; commutator; Lie algebra; Lie group.

Let G be a real Lie group, and let H and K be connected, Lie subgroups, with Lie algebras $\mathfrak{h}$ and $\mathfrak{k}$. The group $[H, K]$ generated by commutators $\{hkh^{-1}k^{-1}|h\epsilon H, k\epsilon K\}$ is arc-wise connected. This implies, by a theorem of Yamabe [1],[1] that $[H, K]$ is a Lie subgroup. The question we raise in this note is: What is the Lie algebra of $[H, K]$? We shall prove that this is the smallest algebra which contains $[\mathfrak{h}, \mathfrak{k}]$ and is invariant under $ad\mathfrak{h}$ and $ad\mathfrak{k}$ (i.e., it is the smallest ideal in the algebra generated by $\mathfrak{h}$ and $\mathfrak{k}$ which contains $[\mathfrak{h}, \mathfrak{k}]$.) An immediate consequence is that if H and K are complex groups, then $[H, K]$ is also complex. Of course, one obtains as a special case the known fact that if H and K are normal, then $[\mathfrak{h}, \mathfrak{k}]$ is the Lie algebra of $[H, K]$. More generally, the Lie algebra of $[H, K]$ is the smallest algebra $\mathfrak{m}$ containing $[\mathfrak{h}, \mathfrak{k}]$, if and only if $\mathfrak{m}$ is invariant under $ad\mathfrak{h}$ and $ad\mathfrak{k}$.

We shall first give a list of notation and terminology. In the following, G is a real Lie group, with Lie algebra $\mathfrak{g}$.

(1) If $g\epsilon G$ and X is a tangent vector or vector field on G, then gX and Xg denote the left and right translation of X. (i.e., if l_g, r_g are the left and right translations of G, then $gX = dl_g(X)$ and $Xg = dr_g(X)$.)

(2) We denote by ad and Ad the adjoint representations of $\mathfrak{g}$ and G, respectively. Thus $ad(X)Y = [X, Y]$ and $Ad(g)Y = gYg^{-1}$. $\theta(g)$ denotes the inner automorphism $x \to gxg^{-1}$. Note that

$$Ad(\exp X) = \exp(adX) \text{ and } \theta(g)\exp X = \exp[Ad(g)X].$$

(3) If X is a tangent vector at some point of G, $\hat{X}$ denotes the corresponding left-invariant vector field. Of course, $\mathfrak{g}$ is the algebra of left-invariant vector fields on G.

(4) If X is a vector field, $g\epsilon G$, then X_g is the value of X at g. If $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra, $\mathfrak{h}_g = \{X_g | X\epsilon \mathfrak{h}\}$.

(5) If $\mathfrak{h}$ and $\mathfrak{k}$ are subalgebras of $\mathfrak{g}$, $[\mathfrak{h}, \mathfrak{k}]$ denotes the linear span of the commutators $\{[X, Y] | X\epsilon \mathfrak{h}, Y\epsilon \mathfrak{k}\}$.

(6) If $\mathfrak{m}$ is a subspace of $\mathfrak{g}$, $\mathscr{A}\mathfrak{m}$ denotes the subalgebra generated by $\mathfrak{m}$.

(7) If $\mathfrak{m}$ is a subspace, $\mathfrak{h}$ a subalgebra, with $\mathfrak{m} \subset \mathfrak{h}$, then $\mathfrak{I}(\mathfrak{m}, \mathfrak{h})$ denotes the smallest ideal in $\mathfrak{h}$ which contains $\mathfrak{m}$. Thus $\mathfrak{I}(\mathfrak{m}, \mathfrak{h})$ is the linear span of $\mathfrak{m}$ and elements of the form $(adX_1)(adX_2) \ldots (adX_r)Y$, where $X_i\epsilon \mathfrak{h}$, $Y\epsilon \mathfrak{m}$. We write $\mathfrak{I}[\mathfrak{h}, \mathfrak{k}]$ for $\mathfrak{I}([\mathfrak{h}, \mathfrak{k}], \mathscr{A}(\mathfrak{h}+\mathfrak{k}))$.

(8) If $g: I \to G$ is a differentiable curve (where I is an interval), we write $\left.\frac{dg}{dt}\right|_{t_0}$ for the tangent vector $dg\left|_{t_0}\left(\frac{d}{dt}\right)\right.$. We shall write $0((t-t_0)^k)$ for a function $X(t)$ (with values in $\mathfrak{g}$) if there exist numbers $M > 0$, $\epsilon > 0$ so that $||X(t)|| < M|t-t_0|^k$ when $|t-t_0| < \epsilon$. (Here $||X||$ is some norm in $\mathfrak{g}$.)

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For example, if $g(t)$ is a differentiable curve in G, with $g(t_0)=1$ and $\left.\frac{dg}{dt}\right|_{t_0}=X$, then (for t near t_0) $\log[g(t)]=(t-t_0)X+0((t-t_0)^2)$, or equivalently, $g(t)=\exp[(t-t_0)X+0((t-t_0)^2)]$. The Campbell-Baker-Hausdorff formula implies.

$$\exp(tX)\exp(tY)=\exp\left(t(X+Y)+\frac{t^2}{2}[X,Y]+0(t^3)\right).$$

LEMMA 1: *Let* H *and* K *be connected Lie subgroups of* G, *with Lie algebras* $\mathfrak{h}$ *and* $\mathfrak{k}$.
(a) *If* $h\in H$ *and* $Y\in\mathfrak{k}$, *then* $(\mathrm{Ad}(h)-1)Y\in\Im[\mathfrak{h},\mathfrak{k}]$.
(b) *If* $k\in K$ *and* $X\in\mathfrak{h}$, *then* $(\mathrm{Ad}(k)-1)X\in\Im[\mathfrak{h},\mathfrak{k}]$.

PROOF: We shall prove (a). Note that $(Ad(h_1h_2)-1)Y=Ad(h_1)(Ad(h_2)-1)Y+(Ad(h_1)-1)Y$. Thus, if the statement is true for h_1 and h_2, it is true for h_1h_2. Therefore it suffices to prove it for a set of generators. We shall show that it is true for $h=\exp(X), X\in\mathfrak{h}$.

$$Ad(\exp X)-1=\exp(adX)-1=\sum_{n=1}^{\infty}\frac{(adX)^n}{n!},$$

while $(adX)^nY\in\Im[\mathfrak{h},\mathfrak{k}]$. *Q.E.D.*

LEMMA 2: *Let* H *and* K *be connected, Lie subgroups of* G, *with Lie algebras* $\mathfrak{h}$ *and* $\mathfrak{k}$. *Let* h(t) (*respectively* k(t)) *be a differentiable curve in* H (*respectively* K) *with* h(0)=1 (*respectively* k(0)=1). *Let* $g(t)=h(\sqrt{t})k(\sqrt{t})h(\sqrt{t})^{-1}k(\sqrt{t})^{-1}$. *Then for each* $t_0\geq 0$ *at which* g(t) *is defined,*

$$\left.\frac{dg}{dt}\right|_{t_0}\in\Im[\mathfrak{h},\mathfrak{k}]_{g(t_0)}.$$

PROOF: For $t_0=0$, $\left.\frac{dg}{dt}\right|_0=[\hat{X},\hat{Y}]_1$, where $X=\left.\frac{dh}{dt}\right|_0$ and $Y=\left.\frac{dk}{dt}\right|_0$. We must find out what happens when $t_0>0$.

Let $u=\sqrt{t}$, $u_0=\sqrt{t_0}$, $h_0=h(u_0)$, $k_0=k(u_0)$,

$$g_0=g(t_0)=h_0k_0h_0^{-1}k_0^{-1},$$

$$X=\left.\frac{dh(u)}{du}\right|_{u_0},\ Y=\left.\frac{dk(u)}{du}\right|_{u_0}\text{ and }Z=\left.\frac{dg(t)}{dt}\right|_{t_0}.$$

The translated curve $g_0^{-1}g(t)$ has tangent vector $g_0^{-1}Z$ at 1. We shall use properties of exp to compute this tangent vector.
Noting that

$$\left.\frac{d}{du}(h_0^{-1}h(u))\right|_{u_0}=h_0^{-1}X,$$

and

$$\left.\frac{d}{du}(h_0h(u)^{-1})\right|_{u_0}=-\left.\frac{d}{du}(h(u)ho^{-1})\right|_{u_0}=-Xh_0^{-1}.$$

We obtain

(1) $h_0^{-1}h(u)=\exp\{(u-u_0)\hat{X}+0((u-u_0)^2)\}$,
(2) $h_0h(u)^{-1}=\exp\{-(u-u_0)Ad(h_0)\hat{X}+0((u-u_0)^2)\}$.

Similarly,

(3)* $k(u)k_0^{-1}=\exp\{(u-u_0)Ad(k_0)\hat{Y}+0((u-u_0)^2)\}$,
(4) $k(u)^{-1}k_0=\exp\{-(u-u_0)\hat{Y}+0((u-u_0)^2)\}$.

Next, we see that

$$g_0^{-1}g(t)=k_0h_0k_0^{-1}h_0^{-1}h(u)k(u)h(u)^{-1}k(u)^{-1}$$

$$=\theta(k_0)\{[\theta(h_0k_0^{-1})(h_0^{-1}h(u)k(u)k_0^{-1})][h_0h(u)^{-1}k(u)^{-1}k_0]\}.$$

Now by using equations (1)−(4),

(5) $\exp(sU+0(s^2))\exp(sV+0(s^2))=\exp\{s(U+V)+0(s^2)\}$

and

(6) $\theta(g)\exp(U)=\exp[Ad(g)U]$,

a straightforward computation shows that $g_0^{-1}g(t)=\exp[W(t)]$, where

$$W(t)=(u-u_0)\{Ad(k_0h_0)(Ad(k_0^{-1})-1)\hat{X}+Ad(k_0)(Ad(h_0)-1)\hat{Y}\}+0((u-u_0)^2).$$

It follows that the tangent vector $g_0^{-1}Z$ is the value of the vector field

$$\left.\frac{dW}{dt}\right|_{t_0}=\frac{1}{\sqrt{t_0}}\{Ad(k_0h_0)(Ad(k_0^{-1})-1)\hat{X}+Ad(k_0)(Ad(h_0)-1)\hat{Y}\}$$

at the unit element. Thus, by Lemma 1, $g_0^{-1}Z\epsilon\Im[\mathfrak{h},\mathfrak{k}]_1$ and $Z\epsilon\Im[\mathfrak{h},\mathfrak{k}]_{g_0}$.

Q.E.D.

LEMMA 3: [H, K] *is normalized by* H *and* K.

PROOF: Let h and $h_1\epsilon H$, $k\epsilon K$. Then

$$h_1(hkh^{-1}k^{-1})h_1^{-1}=(h_1hkh^{-1}h_1^{-1}k^{-1})(kh_1k^{-1}h_1^{-1}).$$

This shows that $[H, K]$ is normalized by H. Since $[H, K]=[K, H]$, the same is true for K.

Q.E.D.

THEOREM: *Let* G *be a real Lie group, and let* H *and* K *be connected, Lie subgroups with Lie algebras* $\mathfrak{h}$ *and* $\mathfrak{k}$. *Then* [H, K] *is a Lie subgroup with Lie algebra* $\Im[\mathfrak{h},\mathfrak{k}]$.

PROOF. We have already remarked that Yamabe's theorem implies that $[H, K]$ is a Lie subgroup. Let $\mathfrak{m}$ denote its Lie algebra. We first show that $\Im[\mathfrak{h},\mathfrak{k}]\subset\mathfrak{m}$. If $X\epsilon\mathfrak{h}$, $Y\epsilon\mathfrak{k}$, then the curve

$$g(t)=\exp(\sqrt{t}X)\exp(\sqrt{t}Y)\exp(-\sqrt{t}X)\exp(-\sqrt{t}Y)$$

lies in $[H, K]$. Its tangent vector at 1 is $[X, Y]_1$. This shows that $[\mathfrak{h},\mathfrak{k}]\subset\mathfrak{m}$. Lemma 3 implies that $\mathfrak{m}$ is invariant under $ad(\mathfrak{h})$ and $ad(\mathfrak{k})$. Therefore $\Im[\mathfrak{h},\mathfrak{k}]\subset\mathfrak{m}$.

Now we show that $\mathfrak{m}\subset\Im[\mathfrak{h},\mathfrak{k}]$. Equivalently, we show that $[H, K]\subset L$, where L is the connected Lie subgroup whose Lie algebra is $\Im[\mathfrak{h},\mathfrak{k}]$. Let $h\epsilon H$, $k\epsilon K$, and let $h(t)$ and $k(t)$ be differentiable curves in H and K, respectively, so that $h(0)=1=k(0)$, $h(1)=h$ and $k(1)=k$. Let

$$g(t)=h(\sqrt{t})k(\sqrt{t})h(\sqrt{t})^{-1}k(\sqrt{t})^{-1}.$$

Lemma 2 says that

$$\left.\frac{dg}{dt}\right|_{t_0}\epsilon\Im[\mathfrak{h},\mathfrak{k}]_{g(t_0)}\qquad(0\leq t_0\leq 1).$$

Therefore, the curve $g(t)$ lies in the maximal connected integral manifold (through 1) of the differential system $\Im[\mathfrak{h},\mathfrak{k}]$. In other words, the curve $g(t)$ lies in L. Thus, the commutators $hkh^{-1}k^{-1}\epsilon L$, and consequently $[H, K]\subset L$.

Q.E.D.

COROLLARY 1: *Let* G *be a complex Lie group, and let* H *and* K *be connected (complex) Lie subgroups. Then* [H, K] *is a (complex) Lie subgroup.*

PROOF: Considering G with its real structure, we see that $[H, K]$ is a real Lie subgroup whose Lie algebra is $\Im[\mathfrak{h},\mathfrak{k}]$ (where $\mathfrak{h}$ and $\mathfrak{k}$ are the Lie algebras of H and K). Since $\mathfrak{h}$ and $\mathfrak{k}$ are complex, so is $\Im[\mathfrak{h},\mathfrak{k}]$ and $[H, K]$.

Q.E.D.

COROLLARY 2: *The Lie algebra of* [H, K] *is* $\mathscr{A}[\mathfrak{h},\mathfrak{k}]$, *if and only if* $\mathscr{A}[\mathfrak{h},\mathfrak{k}]$ *is invariant under* ad($\mathfrak{h}$) *and* ad($\mathfrak{k}$).

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On a Class of Nonlinear Transformations and Their Applications to the Evaluation of Infinite Series*

H. L. Gray [1] and W. D. Clark [2]

(March 12, 1969)

In this paper the problem of increasing the rate of convergence of infinite sequences and series is considered by means of a class of nonlinear transformations. The transformations employed are the discrete analogue of the so called G-transformations and the theory surrounding them is investigated in some detail. The theoretical results are demonstrated by numerous examples.

Key words: Aitken transformations; G-transformations; nonlinear transformations; rapidity of convergence.

1. Introduction

In several recent papers [5, 6, 7, 8] [3] H. L. Gray, T. A. Atchison, and others have considered a class of nonlinear transformations, referred to as G-transformations, for the purpose of evaluating improper integrals. The complete motivation for these transformations is given in [7] although the underlying concept which suggested this motivation is contained in an earlier paper [11] by S. Lubkin. However the setting in [7] affords a certain degree of generality not obtainable in [11] and thus suggests a general class of nonlinear transformation which might be of use in the problem of [11], i.e., the problem of increasing the rate of convergence of an infinite series to its limit.

To be more specific. In [7] the G-transformation, or the class of transformations referred to as G-transformations, is defined as follows:

Let f be a continuous function on $[a, \infty)$ and let

$$F(t)=\int_a^t f(x)dx \to S \neq \mp\infty \text{ as } t \to \infty. \tag{1.1}$$

Moreover let

$$R(t, g(t))=\frac{f(t)}{f(g(t))g'(t)}, \tag{1.2}$$

where $g(t) \to \infty$ as $t \to \infty$. Then the G-transformation is defined by

$$G[F; g(t), t]=\frac{F(t)-R(t, g(t))F(g(t))}{1-R(t, g(t))}. \tag{1.3}$$

From the motivation which produced (1.3) $G[F:g(t), t]$ was expected to be of some value in the direct evaluation of improper integrals. As has been demonstrated in [5, 6, 7, 8] this turns out to be the case.

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[3] Figures in brackets indicate the literature references at the end of this paper.

In this paper a discrete version of (1.3) is considered for the purpose of increasing the rate of convergence of an infinite series. It is shown that the e_1 (Shanks) or T (Lubkin) transformation arises in a natural way from this class of transformations. Moreover close consideration is given to the ratio $R[n, g(n)]$ as a guide for selecting the appropriate transformation for a given series.

The important special cases in which $\lim_{n\to\infty} R[n, g(n)] = 0, 1$ are studied in some detail. Several examples are included.

Before formally defining the class of transformations which is the subject of this paper we give the following definitions.

DEFINITION 1.1: *Let* A(n) *and* B(n) *be sequences of real numbers such that* $\lim_{n\to\infty}$ A(n) = A *and* $\lim_{n\to\infty}$ B(n) = B *and suppose*

$$\lim_{n\to\infty} \left|\frac{A(n)-A}{B(n)-B}\right| = L \neq \infty. \tag{1.4}$$

Then we say $A(n)$ converges more rapidly than $B(n)$ if $L=0$ and at the same rate as $B(n)$ if $L \neq 0$.

DEFINITION 1.2: *If*

$$|A(n)-A| < |B(n)-B| \tag{1.5}$$

for all $n \in (a, b)$ *then we say that* A(n) *converges uniformly better than* B(n) *on* (a, b).

The following simple theorem connects these two definitions.

THEOREM 1.1: *If* A(n) *converges more rapidly than* B(n), *then there exists an* N *such that* A(n) *converges uniformly better than* B(n) *on* (N, ∞).

PROOF: The proof is trivial and can be found in [7].

DEFINITION 1.3: *We say a transformation* T *defined on* A(n) *is exact if there exists an* $N < \infty$ *such that* T[A(n)] = L *if* $n > N$, *where* $\lim_{n\to\infty}$ A(n) = L.

2. The T Transformation

DEFINITION 2.1: *Let* g *be a positive, integer valued function defined on the nonnegative integers such that* $\lim_{n\to\infty} g(n) = \infty$.
Also let

$$S(n) = \sum_{k=1}^{n} a(k) \to S \neq \mp\infty \text{ as } n \to \infty, \tag{2.1}$$

and

$$R[n, g(n)] = \frac{a(n)}{a[g(n)][g(n+1)-g(n)]} = \frac{a(n)}{a[g(n)]\,\Delta\, g(n)}. \tag{2.2}$$

Then we define the T *transformation by*

$$T[S(n); g(n)] = \frac{S(n) - R[n, g(n)]S[g(n)]}{1 - R[n, g(n)]}, \quad R[n, g(n)] \neq 1. \tag{2.3}$$

We assume that either $a[g(n)]\Delta g(n) \neq 0$ *or if* $a[g(n)]\Delta g(n) = 0$ *then* $\lim_{x\to n} R[x, g(x)]$ *exists and we define* R[n, g(n)] *to be that limit. Unless stated to the contrary we will not necessarily assume* $\lim_{n\to\infty} R[n, g(n)]$ *exists. Throughout it will be assumed that* S *has an infinite number of nonzero terms.*

THEOREM 2.1: *For every function* g *for which there exists an* $M > 0$ *such that*

$$|1 - R[n, g(n)]| \geq M \tag{2.4}$$

for n *sufficiently large,* $T[S(n); g(n)] \to S$ *as* $n \to \infty$.

PROOF: Since

$$T[S(n); g(n)] - S[g(n)] = \frac{S(n) - S[g(n)]}{1 - R[n, g(n)]} \to 0 \tag{2.5}$$

as $n \to \infty$, then the theorem follows.

COROLLARY 2.1: *If* $\lim_{n\to\infty} R[n, g(n)]$ *exists and is different from* 1, *then* $T[S(n); g(n)] \to S$ as $n \to \infty$.

THEOREM 2.2: *If* $\lim_{n\to\infty} R[n, g(n)] = L(g) \neq 1$, $L(g)$ *finite, and*

$$\frac{S - S(n)}{S - S[g(n)]} \to L(g), \tag{2.6}$$

then $T[S(n); g(n)]$ *converges more rapidly than* $S[g(n)]$.
Further, if $L(g) \neq 1$ or 0, *then* $T[S(n)]$ *converges more rapidly than either* $S[g(n)]$ or $S(n)$ *if and only if* (2.6) holds.

PROOF: Since

$$\frac{S - T[S(n); g(n)]}{S - S(n)} = \frac{1}{1 - R[n, g(n)]} \left\{1 - R[n, g(n)] \frac{S - S[g(n)]}{S - S(n)}\right\} \tag{2.7}$$

and

$$\frac{S - T[S(n); g(n)]}{S - S[g(n)]} = \frac{1}{1 - R[n, g(n)]} \left\{\frac{S - S(n)}{S - S[g(n)]} - R[n, g(n)]\right\}. \tag{2.8}$$

then the result follows.

COROLLARY 2.2: *Under the conditions of Theorem 2.2, there exists an* N *such that* $T[S(n); g(n)]$ *converges uniformly better than* $S(n)$ *or* $S[g(n)]$ *on* (N, ∞).

PROOF: The result follows from Theorem 1.1.

THEOREM 2.3: $T[S(n); g(n)]$ *converges uniformly better than* $S(n)$ *on* (a, b) *if and only if*

$$-2 < \frac{R[n, g(n)]}{1 - R[n, g(n)]} \frac{S[g(n)] - S(n)}{E(n)} < 0 \tag{2.9}$$

on (a, b), *where* $E(n) = \sum_{k=n+1}^{\infty} a(k) = S - S(n)$.

Consider now the following examples.

EXAMPLE 2.1: Let

$$S(m) = \sum_{k=0}^{m} 2^{-k} \to 2 \text{ as } m \to \infty$$

and let $g(m) = m^2$. Then

$$a(m) = 2^{-m},$$

$$a[g(m)] = 2^{-m^2}, \text{ and}$$

$$R[m; g(m)] = \frac{2^{-m}}{2^{-m^2}[(m+1)^2 - m^2]} = \frac{2^{m(m-1)}}{2m+1}. \tag{2.10}$$

Therefore,

$$T[S(m); g(m)] = \frac{\sum_{k=0}^{m} 2^{-k} - \frac{2^{m(m-1)}}{2m+1} \sum_{k=0}^{m^2} 2^{-k}}{1 - \frac{2^{m(m-1)}}{2m+1}}$$

$$= \frac{2m+1}{2m+1-2^{m(m+1)}} \sum_{k=0}^{m} 2^{-k} - \frac{2^{m(m+1)}}{2m+1-2^{m(m+1)}} \sum_{k=0}^{m^2} 2^{-k}. \tag{2.11}$$

For $m=2$,

$$T[S(2); g(2)] \doteq 1.9534 \tag{2.12}$$

which is in error by about 0.0466.
But

$$S(4) = 1.9375 \tag{2.13}$$

which is in error by 0.0625. The use of this particular g has led to only a slightly better approximation. It is also interesting to note that $R[m; g(m)] \to \infty$ as $m \to \infty$.

Now let us consider $g(m) = m-3$ and the same series. Then

$$\begin{aligned} a(m) &= 2^{-m} \\ a[g(m)] &= 8(2^{-m}) \text{ and} \\ R[m; g(m)] &= \frac{1}{8}. \end{aligned} \tag{2.14}$$

Therefore,

$$\begin{aligned} T[S(m); g(m)] &= \frac{\sum_{k=0}^{m} 2^{-k} - \frac{1}{8}\sum_{k=0}^{m-3} 2^{-k}}{1-\frac{1}{8}} \\ &= \frac{8}{7}\sum_{k=0}^{m} 2^{-k} - \frac{1}{7}\sum_{k=0}^{m-3} 2^{-k} \\ &= \frac{1}{7}\left[\sum_{k=0}^{m} 2^{-k+3} - \sum_{k=3}^{m} 2^{-k+3}\right] \\ &= \frac{14}{7} = 2. \end{aligned} \tag{2.15}$$

which is the correct answer for any value of $m \geq 3$.

EXAMPLE 2.2.: Let

$$S(m) = \sum_{k=1}^{m} \frac{(-1)^{k+1}}{k} \to \ln 2 \text{ as } n \to \infty \tag{2.16}$$

and let $g(m) = m-1$. Then

$$\begin{aligned} a(m) &= (-1)^{m+1}/m, \\ a[g(m)] &= (-1)^{m}/(m-1), \text{ and} \\ R[m; g(m)] &= -\frac{m-1}{m} \to -1 \text{ as } m \to \infty. \end{aligned} \tag{2.17}$$

Therefore,

$$T[S(m); g(m)] = \frac{m}{2m-1}\sum_{k=1}^{m} \frac{(-1)^{k+1}}{k} - \frac{1-m}{2m-1}\sum_{k=1}^{m-1} \frac{(-1)^{k+1}}{k}. \tag{2.18}$$

For $m=4$,

$$T[S(4); g(4)] \doteq 0.69048, \tag{2.19}$$

with an error of about 0.00267, whereas

$$S(4) \doteq 0.58333 \tag{2.20}$$

with an error of about 0.10982.

3. The T_{+m} Transformation

From the above two simple examples it is clear that the choice of g can significantly effect the performance of T.

For this reason we will now consider a more specific type of T transformation. We do so by determining g such that $T[S(n); g(n)]$ is exact when $a(n) = r^n$, i.e., such that the transformation is exact on geometric series. In regard to this we have the following theorem.

THEOREM 3.1: *If* $a(n) = r^n$, $|r| < 1$, *then a necessary and sufficient condition for* T *to be exact, i.e.,* $T[S(n); g(n)] \equiv S$, *when n is sufficiently large, is that* $g(n) = n - m > 0$.

PROOF:

$$T[S(n); g(n)] = S + \frac{-\sum_{k=n+1}^{\infty} r^k + R(n) \sum_{k=g(n)+1}^{\infty} r^k}{1 - R(n)}. \tag{3.1}$$

Therefore T is exact if and only if

$$-\sum_{k=n+1}^{\infty} r^k + R(n) \sum_{k=g(n)+1}^{\infty} r^k = -\frac{r^{n+1}}{1-r} + \frac{r^{n+1}}{(1-r)[g(n+1) - g(n)}} = 0 \tag{3.2}$$

when $n > N$. The latter implies that

$$g(n+1) - g(n) = 1 \text{ when } n > N. \tag{3.3}$$

Since g is a positive, integer valued function, (3.3) implies

$$g(n) = n - m > 0. \tag{3.4}$$

Substitution of (3.4) in (3.1) shows the sufficiency.

This leads us to the following important special case which we will set apart by a separate definition.

DEFINITION 3.1.: *We define the* T_{+m} *transformation by*

$$T_{+m}[S(n+m)] \equiv T[S(n+m); n] = \frac{S(n+m) - R(n+m, n)S(n)}{1 - R(n+m, n)}. \tag{3.5}$$

For brevity we will adopt the notation $R(n; m)$ *for* $R(n+m, n)$, *i.e.,*

$$T_{+m}[S(n+m)] = \frac{S(n+m) - R(n; m)S(n)}{1 - R(n; m)}, \tag{3.6}$$

where

$$R(n; m) = \frac{a(n+m)}{a(n)}. \tag{3.7}$$

Some alternative forms of (3.6) will also be used. They are:

$$T_{+m}[S(n+m)] = S(n) + \frac{S(n+m) - S(n)}{1 - R(n; m)}. \tag{3.8}$$

$$T_{+m}[S(n+m)] = S - \frac{1}{1 - R(n; m)}\{S - S(n+m) - R(n; m)[S - S(n)]\} \tag{3.9}$$

and

$$T_{+m}[S(n+m)] = \frac{S(n)S(n+m-1) - S(n+m)S(n-1)}{S(n) - S(n-1) - S(n+m) + S(n+m-1)} \tag{3.10}$$

We can also write, for $n \geq 2$,

$$T_{+m}[S(n+m)] - T_{+m}[S(n-1+m)]$$

$$= [S(n+m) - S(n)]\left[\frac{1}{1-R(n;m)} - \frac{1}{1-R(n-1;m)}\right] = b(n). \tag{3.11}$$

Therefore we have the following partial sum:

$$T_{+m}[S(n+m)] = \sum_{k=1}^{n} b(k), \text{ where} \tag{3.12}$$

$b(1) = T_{+m}[S(1+m)]$ *and* $b(k)$ *is defined by (3.11) for* $k = 2, 3, \ldots$ (3.13)

A case of particular importance is the case where $m = 1$, *although this is by no means the only case of interest here. When* $m = 1$ *we have*

$$T_{+1}[S(n+1)] = \frac{S(n+1) - R(n;1)S(n)}{1 - R(n;1)} \tag{3.14}$$

$$= \frac{S(n+1)S(n-1) - S^2(n)}{S(n+1) + S(n-1) - 2S(n)}, \quad n > 1. \tag{3.15}$$

The transformation defined by (3.14) was first introduced by Aitken in [1]. It was later studied extensively by S. Lubkin in [11] and D. Shanks in [12].

Let us consider the following two examples which justify (1) our interest in $T_{+1}[S(n+1)]$ and (2) our interest in $T_{+m}[S(n+m)]$ when $m \neq 1$.

EXAMPLE 3.1: Consider the Liebnitz expansion for π. That is,

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \ldots . \tag{3.16}$$

From a numerical viewpoint (3.16) is essentially worthless as it stands since it takes more than 4×10^7 terms to obtain eight figure accuracy. It is a remarkable fact, however, that eight figure accuracy can be obtained from only the first 10 terms by means of $T_{+1}[S(n+1)]$. This was demonstrated in [12] by repeated application of T_{+1}. The procedure is indicated below on the first five terms of (3.16).

TABLE 1

n	$S(n)$	T_{+1}	T_{+1}^2
1	4.0000000		
2	2.6666667	3.1666667	
3	3.4666667	3.1333333	3.1421053
4	2.8952381	3.1452381	
5	3.3396825		

EXAMPLE 3.2: Consider now the same sum in the form

$$4 \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n} = \pi \tag{3.17}$$

Thus we have

$$\pi=4+0-\frac{4}{3}+0+\frac{4}{5}+0-\frac{4}{7}+\ldots. \tag{3.18}$$

In this case $T_{+1}[S(n+1)]$ is not worthwhile since $T_{+1}^{k}[S(n+1)]=T_{+1}^{k-1}[S(n+1)]=S(n+1)$ for every k for which the relations are properly defined. Of course we can apply $T_{+1}[S(n+1)]$, as we already have, if we delete the zeros. However in general it may not be a simple matter to determine how to rearrange a series so that $T_{+1}[S(n+1)]$ is useful. For example if we consider the series $\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sin\frac{n\pi}{p}\right)$, then when p is an integer greater than 2 the problem of rearranging the series so as to apply T_{+1} becomes unmanageable due to the difficulty of writing the general term when the zeros are deleted. However the problem is easily handled by selecting $m=p$ in the T_{+m} transformation. Thus a better way to handle the above problem is as follows. Note that

$$\begin{aligned}R(n;\,m)=\frac{a(n+m)}{a(n)}&=\frac{\sin\,[(n+m)\pi/2]}{\sin\,(n\pi/2)}\,\frac{n}{n+m}\\&=\frac{\sin\,(n\pi/2)\,\cos\,(m\pi/2)+\sin\,(m\pi/2)\,\cos\,(n\pi/2)}{\sin\,(\pi/2)}\,\frac{n}{n+m}\\&=[\cos\,(m\pi/2)+\cot\,(n\pi/2)\,\sin\,(m\pi/2)]\frac{n}{n+m}.\end{aligned} \tag{3.19}$$

Thus, if $m=1$, $R(n;\,1)$ does not exist in general. But, if $m=2$, $R(n;\,2)$ exists for every n and we obtain the following table from applying T_{+2} repeatedly to the first ten terms of (3.18). That is, using only the partial sums we used in table 1.

TABLE 2

n	$S(n)$	T_{+2}	T_{+2}^{2}	$T_{+2}^{3}\ldots$
1	4	3.1111111	3.1444444	3.1417363
2	4	3.1666667	3.1425851	
3	2.6666667	3.1466667	3.1404762	
4	2.6666667	3.1333333	3.1413119	
5	3.4666667	3.1401361		
6	3.4666667	3.1452381		
7	2.8952381	3.1421517		
8	2.8952381			
9	3.3396825			
10	3.3396825			

It should be mentioned that the form (3.10) was utilized in constructing table 2 since this simplifies the repeated application.

As already mentioned the results in table 1 were first given in [12]. In that same paper the author remarks that if (3.16) is written in the form (3.18), then $T_{+1}[S(n+1)]$ is not directly applicable. In this example we have seen, however, that T_{+m} is directly applicable when $m=2$.

From the above it is clear that $T_{+m}[S(n+m)]$ bears some investigation in general. This will be the purpose of the next several pages. The case $m=1$ will not be considered separately since it has been extensively studied in the existing literature. It will first be necessary to introduce some results which will be useful in proving these theorems.

THEOREM 3.2: *Let* A(n) *and* B(n) *be sequences such that* $\lim_{n\to\infty}\mathrm{A(n)}=0$ *and* B(n) *is monotone and converges to zero as* $n\to\infty$. *If*

$$\lim_{n\to\infty}\frac{\mathrm{A(n)}-\mathrm{A(n-1)}}{\mathrm{B(n)}-\mathrm{B(n-1)}} \tag{3.20}$$

exists, finite or infinite, then

$$\lim_{n\to\infty}\frac{A(n)}{B(n)}=\lim_{n\to\infty}\frac{A(n)-A(n-1)}{B(n)-B(n-1)}. \tag{3.21}$$

PROOF; See [1].

THEOREM 3.3: *If*

(i) S(n) *is a monotone sequence with limit*

$$S=\sum_{n=1}^{\infty}a(n) \tag{3.22}$$

(ii) m *is a nonnegative integer and*

$$R_m=\lim_{n\to\infty}\frac{a(n+m)}{a(n)}, \tag{3.23}$$

then

$$\lim_{n\to\infty}\frac{S-S(n+m)}{S-S(n)}=R_m. \tag{3.24}$$

PROOF: Since $S(n)$ is a monotone sequence, $S-S(n)$ is a monotone sequence which converges to zero. Thus we can apply Theorem 3.2 and

$$\lim_{n\to\infty}\frac{S-S(n+m)-S+S(n+m-1)}{S-S(n)-S+S(n-1)}=\lim_{n\to\infty}\frac{S(n+m)-S(n+m-1)}{S(n)-S(n-1)}$$

$$=\lim_{n\to\infty}\frac{a(n+m)}{a(n)}$$

$$=\lim_{n\to\infty}\frac{S-S(n+m)}{S-S(n)}. \tag{3.25}$$

It should be noted that Theorem 3.3 can be restated in the form

$$\lim_{n\to\infty}\frac{\sum_{k=n+m+1}^{\infty}a(k)}{\sum_{k=n+1}^{\infty}a(k)}=\lim_{n\to\infty}\frac{a(n+m)}{a(n)}. \tag{3.26}$$

This result is directly analogous to applying L'Hospitals theorem to improper integrals.

A special case of the above theorem arises when the series consists of terms of constant sign when n is sufficiently large. When a series is alternating, $S(n)$ will generally not be monotone for any range of n and the above theorem does not apply. However a result similar to Theorem 3.3 can be obtained when the terms are decreasing in absolute value.

THEOREM 3.4: *If*

$$\text{(i)}\,S=\sum_{n=1}^{\infty}a(n),\ a(n)=(-1)^nC(n),$$

where $C(n)>C(n+1)>0$,

$$\text{(ii)}\,R(n+1)=\frac{a(n+1)}{a(n)}\to R\neq -1,$$

and

(iii) m *is a positive integer,*

then
$$\lim_{n\to\infty}\frac{S-S(n+m)}{S-S(n)}=\lim_{n\to\infty}\frac{a(n+m)}{a(n)}=R^m. \tag{3.27}$$

Moreover, if $R=-1$ and
$$\lim_{n\to\infty}\frac{1+R(n;1)}{1+R(n-1;1)}=1, \tag{3.28}$$

then the above results hold.

PROOF: Consider the case $m=1$. By hypothesis, S converges and the sequences $S(2n)$ and $S(2n+1)$ are decreasing and increasing respectively. Therefore $S(2n)-S$ and $S-S(2n+1)$ decrease to zero. But

$$\frac{[S(2m+1)-S]-[S(2m-1)-S]}{[S(2m)-S]-[S(2m-2)-S]}=\frac{a(2n)+a(2n+1)}{a(2n-1)+a(2n)}$$
$$=R(2m-1,1)\left[\frac{1+R(2m,1)}{1+R(2m-1,1)}\right], \tag{3.29}$$

which converges to R under the conditions stated. Applying Theorem 3.2, it follows that

$$\lim_{n\to\infty}\frac{S-S(2n+1)}{S-S(2n)}=R. \tag{3.30}$$

The same argument can be applied to the quotient $[S-S(2n+2)]/[S-S(2n+1)]$ and since both of these sequences converge to R it follows that

$$\lim_{n\to\infty}\frac{S-S(n+1)}{S-S(n)}=R. \tag{3.31}$$

In the general case we note that

$$\frac{a(n+m)}{a(n)}=\frac{a(n+1)}{a(n)}\cdot\frac{a(n+2)}{a(n+1)}\cdot\frac{a(n+3)}{a(n+2)}\cdot\cdots\cdot\frac{a(n+m)}{a(n+m-1)}, \tag{3.32}$$

and thus,

$$\lim_{n\to\infty}\frac{a(n+m)}{a(n)}=R^m. \tag{3.33}$$

Also

$$\frac{S-S(n+m)}{S-S(n)}=\frac{S-S(n+1)}{S-S(n)}\cdot\frac{S-S(n+2)}{S-S(n+1)}\cdot\cdots\cdot\frac{S-S(n+m)}{S-S(n+m-1)}, \tag{3.34}$$

and referring to the case $m=1$ we see that

$$\lim_{n\to\infty}\frac{S-S(n+m)}{S-S(n)}=R^m, \tag{3.35}$$

and the theorem follows.

THEOREM 3.5: *For every* m *for which there exists an* $M>0$ *such that* $|1-R(n;m)|\geq M$ *for* n *sufficiently large,* $T_{+m}[S(n+m)]$ *converges to* S.

PROOF: The result is a special case of Theorem 2.1.

COROLLARY 3.1: *If* $\lim_{n\to\infty} R(n;m)=R_m\neq 1$, *then* $T_{+m}[S(n+m)]$ *converges to* S.

When $m=1$ a stronger theorem than the above can be given. This is the case for many of the following theorems. For a discussion of such theorems when $m=1$ see [11].

A further comment should be made in regard to Theorem 3.5. That is, in general $T_{+m}[S(n+m)]$, is not a regular transformation. Thus $S(n)$ may converge as $n\to\infty$ and $T_{+m}[S(n+m)]$ diverge. Theorem 3.5 shows this can only occur when $R_m=1$ or when R_m fails to exist.

LEMMA 3.1: *If* a(n) *is monotone then* n[S(n+m)−S(n)]→0 *as* n→∞.

PROOF: Since S converges and $a(n)$ is monotone, then $n\cdot a(n)\to 0$. Thus $na(n+p)+pa(n+p)=(n+p)a(n+p)\to 0$ for each integer $p\geq 0$ and since $a(n+p)\to 0$ as $n\to\infty$ it follows that $na(n+p)\to 0$ as $n\to\infty$. Hence $n[S(n+m)-S(n)]=na(n+1)+\ldots+na(n+m)\to 0$ as $n\to\infty$.

THEOREM 3.6: *If* a(n) *is monotone and* $\lim_{n\to\infty} T_{+m}[S(n+m)]=L(m)$, *then* L(m)=S.

PROOF: Suppose the theorem is false. Then

$$T_{+m}[S(n+m)-S(n)]=\frac{S(n+m)-S(n)}{1-R(n;\,m)}\to L(m)-S\neq 0 \tag{3.36}$$

as $n\to\infty$. Hence

$$\frac{n[S(n+m)-S(n)]}{n[1-R(n;\,m)]}\to L(m)-S\neq 0 \text{ as } n\to\infty. \tag{3.37}$$

But, by lemma 3.1, $n[S(n+m)-S(n)]\to 0$ as $n\to\infty$ and hence we must have $n[1-R(n;m)]\to 0$ as $n\to\infty$.

Now if we let $b(n)=a(nm)$, then

$$\frac{b(n+1)}{b(n)}=\frac{a(nm+m)}{a(nm)}=R(nm;\,m)>0,$$

and thus $nm[1-R(nm;\,m)]\to 0$ as $n\to\infty$ or

$$n[1-R(nm;\,m)]=n\left[1-\frac{b(n+1)}{b(n)}\right]\to 0$$

as $n\to\infty$. Therefore, by Raabe's test, the series

$$\sum_{n=1}^{\infty} b(n)=\sum_{n=1}^{\infty} a(nm) \tag{3.39}$$

diverges. But the latter is false since $S(n)$ converges absolutely as $n\to\infty$ and $\sum_{n=1}^{\infty} b(n)$ *is a subseries of* S. The theorem therefore follows.

LEMMA 3.2: *If* $a(n)=0(n^{-\alpha})$, $\alpha>1$, *as* n→∞, *then* n[S(n+m)−S(n)]→0 *as* n→∞.

PROOF: By hypothesis,

$$|n[S(n+m)-S(n)]|\leq n|a(n+1)|+n|a(n+2)|+\ldots+n|a(n+m)|$$

$$\leq nmMn^{-\alpha}=mMn^{-\alpha+1} \tag{3.40}$$

for some $M>0$, provided n is sufficiently large. The lemma then clearly follows from (3.40).

THEOREM 3.7: *If* $a(n)=0(n^{-\alpha})$, $\alpha>1$, *as* n→∞ *and* $T_{+m}[S(n+m)]$ *converges, then* $T_{+m}[S(n+m)]$ *converges to* S *as* n→∞.

PROOF: The proof is the same as in Theorem 3.6 with the exception that (3.39) converges because $a(n)=0(n^{-\alpha})$.

THEOREM 3.8: *If* $\lim_{n\to\infty} R(n;\,m)=R_m\neq 0,\,1$, *then* $T_{+m}[S(n+m)]$ *converges more rapidly than* S(n+m) *if and only if*

$$\lim_{n\to\infty}\frac{S-S(n+m)}{S-S(n)}=R_m. \tag{3.41}$$

PROOF: The result follows from Theorem 2.2.

COROLLARY 3.2: *If* a(n) *is of constant sign for* n *sufficiently large and* R_m *exists but* $R_m\neq 0,\,1$, *then* $T_{+m}[S(n+m)]$ *converges more rapidly than* S(n+m).

PROOF: The corollary follows from Theorems 3.8 and 3.3.

COROLLARY 3.3: *If* $a(n)=(-1)^n C(n)$, $C(n)>C(n+1)>0$ *and* R_m *exists but* $R_m\neq 0,-1$, *then* $T_{+m}[S(n+m)]$ *converges more rapidly than* S(n+m).

PROOF: The corollary follows from Theorems 3.8 and 3.4.

In the above theorems we have established conditions under which (1) $T_{+m}[S(n+m)] \to S$ and (2) $T_{+m}[S(n+m)] \to S$ more rapidly than $S(n+m)$ as $n \to \infty$. In general our theorems concerning the latter were less general than those concerning the former. Thus in Theorems 3.5 and 3.6 we did not find it necessary to require even the existence of the $\lim_{n\to\infty} R(n;\ m)$. On the other hand, in Theorem 3.8, we required not only the existence of $\lim_{n\to\infty} R(n;\ m)$, but that it be different from 0 or 1. In the case $R_m = 0$, $T_{+m}[S(n+m)]$ may or may not converge more rapidly than $S(n+m)$. This is best seen by the following examples.

EXAMPLE 3.3: Let $S(n) = 1/n!$. Then $S(n) \to 0$ and $S(n+1)/S(n) \to 0$ as $n \to \infty$. Hence $\lim_{n\to\infty} R(n;\ 1) = 0$ by Theorem 3.3. Thus we see in this case that $R_1 = 0$ and $S(n+1)$ converges more rapidly than $S(n)$. We have included this example to emphasize that our interest when $R_1 = 0$ should be centered on comparing the rate of convergence of $T_{+m}[S(n+m)]$ with $S(n+m)$ rather than $S(n)$, since in this case $S(n+m)$ and $S(n)$ do not necessarily converge at the same rate. Note that when either Theorem 3.3 or Theorem 3.4 hold that $S(n+m)$ converges more rapidly than $S(n)$ if $R_1 = 0$ and at the same rate as $S(n)$ if $R_1 \neq 0$.

EXAMPLE 3.4: Let $a(n) = 1/n!$ and $m = 1$. Then $R_1 = 0$ and since $A(n) = S - S(n+1)$ is a deceasing sequence converging to 0, we can apply Theorem 3.2 to $A(n)$ and $B(n) = S - T_{+m}[S(n+m)]$. Thus, using (3.11), we have

$$\frac{B(n)-B(n-1)}{A(n)-A(n-1)} = \frac{b(n)}{a(n+1)} = \frac{a(n)}{a(n+1)}\left[\frac{R(n;\ 1)-R(n-1;\ 1)}{(1-R(n;\ 1))(1-R(n-1;\ 1))}\right]$$

$$= \left[\frac{1-\dfrac{R(n-1;\ 1)}{R(n;\ 1)}}{(1-R(n;\ 1))(1-R(n-1;\ 1))}\right]$$

$$= \frac{1-\dfrac{n+1}{n}}{(1-R(n;\ 1))(1-R(n-1;\ 1))} \tag{3.42}$$

as $n \to \infty$.
Hence

$$\lim_{n\to\infty} \frac{S - T_{+1}[S(n+1)]}{S - S(n+1)} = 0 \tag{3.43}$$

and $T_{+1}[S(n+1)]$ converges more rapidly than $S(n+1)$.

EXAMPLE 3.5: Let $m = 1$ and

$$a(n) = \frac{2}{(n+1)!\,2^{(n^2+3n-4)/2}}$$

Then

$$R(n;\ 1) = \frac{a(n+1)}{a(n)} = \frac{1}{(n+2)2^{n+2}},$$

and therefore $R_1 = 0$.
Moreover

$$\frac{R(n-1;\ 1)}{R(n;\ 1)} = \frac{(n+2)2^{n+2}}{(n+1)2^{n+1}} \to 2 \tag{3.44}$$

as $n \to \infty$.

Therefore, using the notation of the previous example, we have

$$\frac{B(n)-B(n-1)}{A(n)-A(n-1)} = \frac{1-\dfrac{R(m-1;\ 1)}{R(m;\ 1)}}{(1-R(m;\ 1))(1-R(m-1;\ 1))} \to -1 \tag{3.45}$$

as $n \to \infty$.

Hence,

$$\lim_{n\to\infty}\frac{S-T_{+1}[S(n+1)]}{S-S(n+1)}=-1, \tag{3.46}$$

and $T_{+1}[S(n+1)]$ and $S(n+1)$ converge with the same order of rapidity. Note that in the same manner one has

$$\lim_{n\to\infty}\frac{S-T_{+1}[S(n+1)]}{S-S(n)}=\lim_{n\to\infty}\frac{R(n;\,1)-R(n-1;\,1)}{(1-R(n;\,1))(1-R(n-1;\,1))}=0, \tag{3.47}$$

so that $T_{+1}[S(n+1)]$ does converge more rapidly than $S(n)$ in this case.

EXAMPLE 3.6: Let $m=1$ and

$$a(n)=\frac{1}{n!2^{n^3}}\cdot \tag{3.48}$$

Then

$$R(n;\,1)=\frac{1}{(n+1)2^{3n^2+3n+1}}\to 0 \tag{3.49}$$

as $n\to\infty$ and by Theorem 3.3

$$\frac{S-S(n+1)}{S-S(n)}=1-\frac{a(n+1)}{S-S(n)}\to 0 \tag{3.50}$$

as $n\to\infty$.
Moreover

$$\frac{R(n+1;\,1)}{R(n;\,1)}=\left(\frac{n+1}{n+2}\right)\left(\frac{1}{2^{6n\ 6}}\right)\to 0 \tag{3.51}$$

as $n\to\infty$.
But

$$\frac{S-S(n+1)}{S-T_{+1}[S(n+1)]}=\frac{1}{1-\left[\frac{a(n+2)}{S-S(n+1)}\right]\left[\frac{1}{1-R(n;\,1)}\right]\left[\frac{R(n;\,1)}{R(n+1;\,1)}\right]}, \tag{3.52}$$

and thus (3.52) has the limit zero as $n\to\infty$ so that $S(n+1)$ converges more rapidly than $T_{+1}[S(n+1)]$.

In the previous examples we have seen that if $R_m=0$, then the T_{+m} transform may or may not be useful. We will therefore now embark on a series of theorems which are useful in ascertaining the value of T_{+m} in that situation. We begin by listing a sequence of lemmas most of which are obvious or very easily proven.

LEMMA 3.3: *If* R(n; m) *converges to* R_m, *then* $|R_m|\leq 1$.

PROOF: Suppose the lemma is false. Then $|R_m|>1$ and therefore there exists an N such that $|R(n;\,m)|>1$ where $n\geq N$, i.e., $|a(n+m)|>|a(n)|$. Thus, $\lim_{n\to\infty}a(n)\neq 0$, so that $S(n)$ does not converge as $n\to\infty$. This is a contradiction and the theorem follows.

LEMMA 3.4: *If* R(n; 1)→ R, *then* R(n; m)→ R^m *as* n→ ∞, *i.e., if* $R_1=R$ *then* $R_m=R^m$.

PROOF: The result follows easily by noting that

$$R(n;\,m)=\frac{a(n+m)}{a(n)}=\frac{a(n+1)}{a(n)}\cdot\frac{a(n+2)}{a(n+1)}\cdot\cdot\cdot\frac{a(n+m)}{a(n+m-1)}$$

$$=R(n;\,1)\cdot R(n+1;\,1)\cdot\cdot\cdot R(n+m;\,1). \tag{3.53}$$

LEMMA 3.5:

$$R(n;\,m)=\frac{a(n+m)}{a(n)}$$

$$=\frac{a(n+m+1)}{a(n+1)}\cdot\frac{a(n+m)}{a(n+m+1)}\cdot\frac{a(n+1)}{a(n)}$$

$$=\frac{R(n+1;\,m)}{R(n+m;\,1)}R(n;\,1) \tag{3.54}$$

LEMMA 3.6:

$$\frac{R(n;\, m)}{R(n+1;\, m)} \to A \text{ as } n \to \infty \tag{3.55}$$

if and only if

$$\lim_{n\to\infty} \frac{R(n;\, 1)}{R(n+m;\, 1)} = A. \tag{3.56}$$

PROOF: This lemma follows immediately from Lemma 3.5.

LEMMA 3.7: *If*

$$\lim_{n\to\infty} \frac{R(n;\, 1)}{R(n+m;\, 1)} \neq 1 \tag{3.57}$$

and

$$\lim_{n\to\infty} R(n;\, m) = R_m,\ R_m \text{ finite}, \tag{3.58}$$

then $R_m = 0$.

PROOF: The lemma is obvious.

LEMMA 3.8: *If* $a(n)$ *is of constant sign and*

$$\lim_{n\to\infty} \frac{R(n;\, 1)}{R(n+m;\, 1)} = A \neq 1, \tag{3.59}$$

then

$$\lim_{n\to\infty} R(n;\, 1) = 0, \tag{3.60}$$

where A is finite or infinite.

PROOF: Since $A(n)$ is of constant sign, it follows that $R(n;\, 1) > 0$. Also

$$\frac{R(n;\, 1)}{R(n+m;\, 1)} > 1 \tag{3.61}$$

or

$$\frac{R(n;\, 1)}{R(n+m;\, 1)} < 1 \tag{3.62}$$

for n sufficiently large. However (3.62) and (3.59) imply that $S(n)$ diverges as $n \to \infty$ and hence cannot hold. Thus, for n sufficiently large, say $n \geq N$, we have

$$\left.\begin{aligned} &R(n;\, 1) > R(n+m;\, 1) > R(n+2m;\, 1) > \ldots > 0 \\ &R(n+1;\, 1) > R(n+m+1;\, 1) > \ldots > \ldots > 0 \\ &\cdot \\ &\cdot \\ &\cdot \\ &R(n+m-1) > R(n+2m-1;\, 1) > \ldots > 0, \end{aligned}\right\} \tag{3.63}$$

and since $A \neq 1$, each subsequence in (3.63) converges to 0. Thus $R(n;\, 1) \to 0$ as $n \to \infty$.

LEMMA 3.9: If $a(n) = (-1)^n C(n)$, $C(n) > C(n+1) > 0$, and $\lim_{n\to\infty} \frac{R(n;\, 1)}{R(n+m;\, 1)} = A \neq 1$, then $\lim_{n\to\infty} R(n;\, 1) = 0$.

PROOF: The proof is similar to that of Lemma 3.8.

LEMMA 3.10:

$$\begin{aligned} S(n+m) - S(n) &= a(n+1)[1 + R(n+1;\, 1) + R(n+1;\, 1)R(n+2;\, 1) + \ldots \\ &\quad + R(n+1;\, 1) \ldots R(n+m-1;\, 1)] \\ &= a(n+1)[1 + R(n+1;\, 1) + \ldots + R(n+1;\, m-1)] \end{aligned}$$

PROOF: The result follows by noting that

$$a(n+1)\left[1+\frac{a(n+2)}{a(n+1)}+\ldots+\frac{a(n+m)}{a(n+1)}\right]$$

$$=a(n+1)\left[1+\frac{a(n+2)}{a(n+1)}+\frac{a(n+3)a(n+2)}{a(n+2)a(n+1)}+\ldots+\frac{a(n+m)}{a(n+m-1)}\right.$$

$$\left.\cdot\frac{a(n+m-1)}{a(n+m-2)}\cdot\ldots\frac{a(n+2)}{a(n+1)}\right]. \tag{3.64}$$

LEMMA 3.11: *If* $\mathrm{R(n;\ 1)}$ *is bounded by* M, *then* $\dfrac{\mathrm{S(n+m)-S(n)}}{\mathrm{a(n+1)}}$ *is bounded by* $\dfrac{1-\mathrm{M^m}}{1-\mathrm{M}}$ *if* $\mathrm{M}\neq 1$ *and by* m *if* $\mathrm{M}=1$.

PROOF: The proof follows immediately from Lemma 3.10.

Note that if R_1 exists, then it is finite by Lemma 3.3 and hence $R(n;\ 1)$ is bounded.

LEMMA 3.12

$$\frac{S(n+m)-S(n)}{S(n+2m)-S(n+m)}=\frac{1}{R(n+1;\ m)}\left[\frac{1+\sum_{k=1}^{m-1}\prod_{i=1}^{k}R(n+i;\ 1)}{1+\sum_{k=1}^{m-1}\prod_{i=1}^{k}R(n+m+i;\ 1)}\right]. \tag{3.55}$$

PROOF: The lemma follows immediately from Lemma 3.10.

LEMMA 3.13:

$$\frac{S-S(n+m)}{S-S(n)}=\frac{S-S(n)-S(n+m)+S(n)}{S-S(n)}=1-\frac{S(n+m)-S(n)}{S-S(n)}. \tag{3.66}$$

THEOREM 3.9: *If either* $\mathrm{R_1}$ *exists or* $\mathrm{R(n;\ 1)}$ *is bounded,* $\mathrm{S(n)}$ *is monotone, and if there exists* p *and* m *such that* $\lim_{n\to\infty}\dfrac{\mathrm{R(n;\ 1)}}{\mathrm{R(n+m;\ 1)}}=1$ *and* $|1-\mathrm{R(n;\ m)}|\geq \mathrm{p}>0$ *for* n *sufficiently large, then* $\mathrm{T_{+m}[S(n+m)]}$ *converges more rapidly than* $\mathrm{S(n+m)}$.

PROOF: By Lemma 3.11, $\dfrac{S(n+m)-S(n)}{a(n+1)}$ is bounded. Moreover, $S(n)$ is monotone, so that $S-S(n+m)$ is monotone. Now let $B(n)=S-T_{+m}[S(n+m)]$ and $A(n)=S-S(n+m)$. Then

$$\frac{B(n+1)-B(n)}{A(n+1)-A(n)}=\frac{b(n+1)}{a(n+m+1)}$$

$$=\frac{S(n+m)-S(n)}{a(n+m+1)}\left\{\frac{R(n+1;\ m)-R(n;\ m)}{[1-R(n+1;\ m)][1-R(n;\ m)]}\right\}$$

$$=\frac{S(n+m)-S(n)}{a(n+1)}\left\{\frac{1-\dfrac{R(n;\ m)}{R(n+1;\ m)}}{[1-R(n+1;\ m)][1-R(n;\ m)]}\right\} \tag{3.67}$$

and by Lemma 3.6, the latter quantity goes to zero as $n\to\infty$.

Hence,

$$\lim_{n\to\infty}\frac{S-T_{+m}[S(n+m)]}{S-S(n+m)}=0 \tag{3.68}$$

and the theorem follows. One should note that if $m=1$, then the assumption that $R(n;\ 1)$ is bounded can be dropped. This follows from (3.67).

Also note that the main difference between this theorem and Theorem 3.8 and its corollaries is that here we have not required $R_m\neq 0$. In fact we have not even required the existence of R_m.

COROLLARY 3.4: *If* $\mathrm{a(n)}$ *is of constant sign,* $\mathrm{R_1}$ *exists,* $\mathrm{R_1}\neq 1$, *and if* $\lim_{n\to\infty}\dfrac{\mathrm{R(n;\ 1)}}{\mathrm{R(n+m;\ 1)}}=1$, *then* $\mathrm{T_{+m}[S(n+m)]}$ *converges more rapdily than* $\mathrm{S(n+m)}$ *as* $\mathrm{n}\to\infty$.

PROOF: The result is an immediate consequence of Theorem 3.9 by noting that $S(n)$ is eventually monotone and $R(n;\, m) \to R_1^m \neq 1$.

Note that since $R_1 = 0$ implies $R_m = 0$, then when corollary 3.4 applies $T_{+m}[S(n+m)]$ converges more rapidly than $S(n+m)$ when $R_m = 0$ if $\lim_{n\to\infty} \dfrac{R(n;\, 1)}{R(n+m;\, 1)} = 1$. Another useful theorem in this respect is the following.

THEOREM 3.10: *If* $\mathrm{a(n)} = (-1)^n \mathrm{C(n)}$, $\mathrm{C(n)} > \mathrm{C(n+1)} > 0$, $\mathrm{R_1} \neq -1$ *exists and* $\lim_{n\to\infty} \dfrac{\mathrm{R(n;\, 1)}}{R(n+m;\, 1)} = 1$, *then* $\mathrm{T_{+m}[S(n+m)]}$ *converges more rapidly than* $\mathrm{S(n+m)}$.

PROOF: The sequence $S - S(2n+m)$ is monotone and $S - T_{+m}[S(2n+m)]$ converges to 0 since $R_m \neq 1$. Moreover

$$\frac{T_{+m}[S(2n+m)] - T_{+m}[S(2n-2+m)]}{S(2n+m) - S(2n-2+m)} = \frac{b(2n-1) + b(2n)}{a(2n-1+m) + a(2n)}$$

$$= \frac{S(2n-2+m) - S(2n-2)}{a(2n-1+m) + a(2n+m)} \left\{ \frac{R(2n-1;\, m) - R(2n-2;\, m)}{[1 - R(2n-1;\, m)][1 - R(2n-2;\, m)]} \right\}$$

$$+ \frac{S(2n-1+m) - S(2n-1)}{a(2n-1+m) + a(2n+m)} \left\{ \frac{R(2n;\, m) - R(2n-1;\, m)}{[1 - R(2n;\, m)][1 - R(n-1;\, m)]} \right\}$$

$$= \frac{S(2n-2+m) - S(2n-2)}{a(2n-1)} \left\{ \frac{1 - \dfrac{R(2n-2;\, m)}{R(2n-1;\, m)}}{[1 + R(2n+m-1;\, 1)][1 - R(2n;\, m)][1 - R(2n-1;\, m)]} \right\} \tag{3.69}$$

Now by Lemma 3.4, $R(n;\, m) \to R_1^m \neq 1, -1$ as $n \to \infty$ and therefore by Lemma 3.11 $\dfrac{S(n+m) - S(n)}{a(n+1)}$ is bounded. Thus by Lemma 3.6, (3.69) converges to zero.
Hence by Theorem 3.2

$$\lim_{n\to\infty} \frac{S - T_{+m}[S(2n+1+m)]}{S - S(n+m)} = 0. \tag{3.70}$$

A similar argument may be applied for the quotient

$$\frac{S - T_{+m}[S(2n+1+m)]}{S - S(2n+1+m)}.$$

Thus

$$\lim_{n\to\infty} \frac{S - T_{+m}[S(2n+1+m)]}{S - S(n+m)} = 0, \tag{3.71}$$

and the theorem follows.

In the previous two theorems we have shown that for series of constant sign or certain alternating series that a sufficient condition that $T_{+m}[S(n+m)]$ converge more rapidly than $S(n+m)$ when $\lim_{n\to\infty} R(n;\, 1) = 0$ is that $\lim_{n\to\infty} \dfrac{R(n;\, 1)}{R(n+m;\, 1)} = 1$. We will now show that the condition is necessary for these types of series.

THEOREM 3.11: *If* $\mathrm{a(n)}$ *is of constant sign and* $\lim_{n\to\infty} \dfrac{\mathrm{R(n;\, 1)}}{\mathrm{R(n+m;\, 1)}} = \mathrm{A} \neq 1$, *A finite, then* $\mathrm{T_{+m}[S(n+m)]}$ *converges with the same order of rapidity as* $\mathrm{S(n+m)}$.

PROOF: By Lemma 3.8 $R(n;\, 1) \to 0$ as $n \to \infty$ and thus $R(n;\, m) \to 0$ as $n \to \infty$. It follows by Lemma 3.10 that $\lim_{n\to\infty} \dfrac{S(n+m) - S(n)}{a(n+1)} = 1$ and the rest of the proof follows as in Theorem 3.9 except that in this case

$$\lim_{n\to\infty} \frac{S - T_{+m}[S(n+m)]}{S - S(n+m)} = 1 - A \neq 0. \tag{3.72}$$

THEOREM 3.12: *If* $a(n)=(-1)^n C(n)$, $C(n)>C(n+1)>0$, *and*

$$\lim_{n\to\infty}\frac{R(n;\,1)}{R(n+m;\,1)}=A\neq 1,\ A\ \textit{finite},$$

then $T_{+m}[S(n+m)]$ *converges with the same order of rapidity as* $S(n+m)$.

PROOF: The proof is much the same as the previous theorem.

THEOREM 3.13: *If* $a(n)$ *is of constant sign and* $\lim_{n\to\infty}\frac{R(n;\,1)}{R(n+m;\,1)}=\infty$, *then* $S(n+m)$ *converges more rapidly than* $T_{+m}[S(n+m)]$.

PROOF: The proof follows in the same manner as Theorem 3.9.

THEOREM 3.14: *If* $a(n)=(-1)^n C(n)$, $C(n)>C(n+1)>0$ *and* $\frac{R(n;\,1)}{R(n+m;\,1)}\to\infty$ as $n\to\infty$, *then* $S(n+m)$ *converges more rapidly than* $T_{+m}[S(n+m)]$.

COROLLARY 3.5: *If* $|a(n)|$ *is monotone decreasing or* $a(n)$ *is of constant sign and if* $|R_n|\neq 1$ *and* $\lim_{n\to\infty}\frac{R(n;\,1)}{R(n+m;\,1)}$ *exists finite or infinite, then a necessary and sufficient condition that* $T_{+m}[S(n+m)]$ *converge more rapidly than* $S(n+m)$ *is that* $\lim_{n\to\infty}\frac{R(n;\,1)}{R(n+m;\,1)}=1$.

In the above theorems we have established a number of conditions under which $T_{+m}[S(n+m)]$ will converge more rapidly than $S(n+m)$. In general we have seen that it is desirable to select m, if possible, so that $|R_m|<1$, where $R_m=\lim_{n\to\infty}R(n;\,m)$. Thus one should try to select m so that $|a(n+m)|<|a(n)|$ for n sufficiently large and R_m exists. Of course this will not guarantee that $|R_m|<1$, but if m is not chosen in this manner there is no chance that $|R_m|<1$. In fact, by Lemma 3.3 if R_m exists and m cannot be chosen so that $|a(n+m)|<|a(n)|$ for n sufficiently large, then $|R_m|=1$.

Before leaving the T_{+m} transformation let us consider one more point. Initially we obtained this transformation by determining g in the T transformation so that T was exact on geometric series. A question which naturally arises is whether T_{+m} is exact on any other kind of series. The answer is effectively in the negative as the next theorem shows.

THEOREM 3.15: T_{+m} *is exact on* $S(n+m)$ *if and only if*

$$\sum_{n=N}^{\infty}a(n)\equiv\sum_{p=N}^{N+m-1}\sum_{k=0}^{\infty}a(p)\,\alpha^k \tag{3.73}$$

for some N, *where* α *is a constant such that* $|\alpha|<1$.

PROOF: Since

$$S-T_{+m}[S(n+m)]=\frac{1}{1-R(n;\,m)}[S-S(n+m)]-R(n;\,m)[S-S(n)], \tag{3.74}$$

it is clear that $T_{+m}[S(n+m)]\equiv S$ for n sufficiently large if and only if there exists an N such that

$$\frac{S-S(n+m)}{S-S(n)}\equiv R(n;\,m) \tag{3.75}$$

for $n\geq N$. But (3.75) is true if and only if $R(n;m)=\alpha$, α a constant, when $n\geq N$. To see the latter note that if $R(n;m)=\alpha$ when $n\geq N$, then

$$\frac{S-S(n+m)}{S-S(n)}=\frac{\sum_{j=n+1}^{\infty}a(j+m)}{\sum_{j=n+1}^{\infty}a(j)}=\alpha\frac{\sum_{j=n+1}^{\infty}a(j)}{\sum_{j=n+1}^{\infty}a(j)}=\alpha=R(n;\,m) \tag{3.76}$$

On the other hand if T_{+m} is exact on $S(n+m)$, then

$$S(n+m)-S=R(n;\,m)[S(n)-S]\ \text{for}\ n\geq N. \tag{3.77}$$

Thus

$$[S(n+m+1)-S]-[S(n+m)-S]=R(n+1;\,m)[S(n+1)-S]-R(n;\,m)[S(n)-S]. \tag{3.78}$$

Hence

$$\begin{aligned} a(n+m+1) &= -S[R(n+1;\,m)-R(n;\,m)]+[R(n+1;\,m)-R(n;\,m)]S(n) \\ &\quad +R(n+1;\,m)[S(n+1)-S(n)] \\ &= [R(n+1;\,m)-R(n;\,m)][S(n)-S]+a(n+m+1). \end{aligned} \tag{3.79}$$

Therefore,

$$S=S(n) \text{ or } R(n+1;\,m)=R(n;\,m) \text{ when } n \geq N.$$

The former is false and consequently $R(n;\,m)=\alpha$ if $n \geq N$. Now since $R(n,\,m)$ is constant when $n \geq N$ and $S(n)$ converges, then $|\alpha|<1$ and T_{+m} exact implies for $k=1, 2, \ldots$ that

$$\begin{aligned} a(n)\,\alpha^k &= a(n+km) \\ a(n+1)\,\alpha^k &= a(n+km+1) \\ &\vdots \\ a(n+m-1)\,\alpha^k &= a(n+km+m-1), \end{aligned} \tag{3.80}$$

and therefore T_{+m} exact implies that

$$\begin{aligned} \sum_{n=N}^{\infty} a(n) &= [a(N)+a(N+1)+\ldots+a(N+m-1)]\sum_{k=0}^{\infty}\alpha^k \\ &= \sum_{p=N}^{N+m-1}\sum_{k=0}^{\infty} a(p)\,\alpha^k \end{aligned} \tag{3.81}$$

and the necessity is proved.

The sufficiency easily follows since $\Sigma\,\alpha^k$ is a geometric series.

4. The $T_{\cdot m}$ Transformation

In the previous section we considered a transformation which we called the T_{+m} transformation. In particular we found the transformation to be quite effective in increasing the rate of convergence of a series except when $R_1=1$. Thus if

$$S(n)=\sum_{k=1}^{n}\frac{1}{k^p}, \qquad p>1 \tag{4.1}$$

we would not be surprised to find that $T_{+m}[S(n+m)]$ does not converge more rapidly than $S(n+m)$ since

$$R_1^m=\lim_{n\to\infty}\frac{a(n+m)}{a(n)}=\lim_{n\to\infty}\frac{n}{(n+m)}=1 \tag{4.2}$$

for every m.

In searching for a transformation which will be effective when $R_1=1$ we return to the T transformation and recall that T_{+m} was obtained by requiring T to be exact on geometric series. Thus we might obtain a transformation which is more effective on series such as (4.1) if we require T to be exact on a series which more closely resembles (4.1) than a geometric series. These considerations lead us to the following theorem.

THEOREM 4.1: *If* $a(n)=(n-1)^{(-x)}$, *then a necessary and sufficient condition for* T *to be exact is that* $g(n)=\frac{n}{\alpha}$, *where* $\frac{n}{\alpha}$ *is a positive integer and* $\alpha \neq 1$. *Before proving the theorem we should point out that the symbol* $n^{(-r)}$ *denotes the fractorial function, i.e.*

$$n^{(-r)}=\frac{1}{(n+1)(n+2)\ \ldots\ (n+r)},\ r\geq 1. \tag{4.3}$$

Thus

$$(n-1)^{(-x)}=\frac{1}{n(n+1)\ \ldots\ (n+x-1)}. \tag{4.4}$$

We will now proceed with the proof of Theorem 4.1.

PROOF: As in Theorem 3.1, the result will follow if and only if

$$-\sum_{k=n+1}^{\infty} a(k)+R(n, g(n))\sum_{k=g(n)+1}^{\infty} a(k)\equiv 0 \tag{4.5}$$

when n is sufficiently large.
But (4.5) can be written

$$\frac{1}{x-1}\,n^{(-x+1)}-R(n, g(n))(g(n))^{(-x+1)}\equiv 0. \tag{4.6}$$

The latter implies that

$$g(n+1)-g(n)=\frac{g(n)}{n}, \tag{4.7}$$

and hence

$$g(n)=\frac{n}{\alpha},$$

where $\alpha \neq 1$ is any constant such that $\frac{n}{\alpha}$ is a positive integer. The above theorem leads us to the following definition.

DEFINITION 4.1. Let m be a positive integer such that $m>1$. Then we define the $T_{\cdot m}$ transformation by

$$T_{\cdot m}[S(nm)]=T[S(nm);\ n]$$

$$=\frac{S(nm)-R(nm, n)S(n)}{1-R(nm, n)}. \tag{4.8}$$

As in T_{+m}, for convenience, we will adopt a slightly different notation for $R(nm, n)$ in this special case. Thus we will let $R(nm, n)=\rho(n; m)$. This notation will be convenient and save confusion with $R(n; m)$ of section 3. Thus

$$T_{\cdot m}[S(nm)]=\frac{S(nm)-\rho(n; m)S(n)}{1-\rho(n; m)}, \tag{4.9}$$

where

$$\rho(n; m)=\frac{ma(nm)}{a(n)}. \tag{4.10}$$

Some alternate forms for (4.9) are the following:

$$T_{\cdot m}[S(nm)]=S(n)+\frac{S(nm)-S(n)}{1-\rho(n; m)} \tag{4.11}$$

$$T_{\cdot m}[S(nm)]=\frac{mS(n)S(nm-1)-S(n-1)S(nm)-(m-1)S(n)S(nm)}{S(n)-S(n-1)-m[S(nm)-S(nm-1)]}. \tag{4.12}$$

THEOREM 4.2: *If there exists an* n *and an* M *such that* $|1-\rho(n, m)| \geq M > 0$ *for* n *sufficiently large, then* $T_{\cdot m}[S(nm)]$ *converges to* S.

PROOF: The result is a special case of Theorem 2.1.

COROLLARY 4.1: *For every* m *such that* $\lim_{n\to\infty} \rho(n; m) = \rho_m \neq 1$, $T_{\cdot m}[S(nm)]$ *converges to* S. *In the future the symbol* ρ_m *will always refer to* $\lim_{n\to\infty} \rho(n; m)$.

LEMMA 4.1: *If* $a(n) = 0(n^{-\alpha})$ *as* $n \to \infty$ *and* $\alpha > 1$, *then*

$$\lim_{n\to\infty} (\log n)(S(nm) - S(n)) = 0. \tag{4.13}$$

PROOF: There exists m and N such that for $n \geq N$

$$\begin{aligned} |(\log n)[S(nm) - S(n)]| &\leq (\log n)(|a(n+1)| + |a(n+2)| + \ldots + |a(nm)|) \\ &\leq (\log n)M[(n+1)^{-\alpha} + (n+2)^{-\alpha} + \ldots + (nm)^{-\alpha}] \\ &\leq (\log n)Mn(m-1)(n+1)^{-\alpha+1} \\ &\leq M(\log n)(m-1)(n+1)^{-\alpha+1}. \end{aligned}$$

But $\alpha > 1$ and $(\log n)(n+1)^{-\alpha+1}$ converges to zero as $n \to \infty$. Thus the lemma follows.

THEOREM 4.3: *If* $a(n) = 0(n^{-\alpha})$, $\alpha > 1$, *then for every* m *such that* $T_{\cdot m}[S(nm)]$ *converges as* $n \to \infty$ *we have*

$$\lim_{n\to\infty} T_{\cdot m}[S(nm)] = \lim_{n\to\infty} S(n) = S. \tag{4.14}$$

PROOF: Let m be any integer such that $m > 1$ and $T_{\cdot m}$ converges to L_m and assume that

$$T_{\cdot m}[S(nm)] - S(n) = \frac{S(mn) - S(n)}{1 - \rho(n; m)} \to L_m - S \neq 0 \tag{4.15}$$

as $n \to \infty$. By Lemma 4.3 $(\log n)(S(mn) - S(n)) \to 0$ as $n \to \infty$. Thus, from (4.15), $(\log n)(1 - \rho(n; m)) \to 0$ as $n \to \infty$. Now let $n = m^p$, $p = 0, 1, 2, 3, \ldots$. Then $\log n = p \log m$ and consequently $p \to \infty$ if and only if $n \to \infty$. Therefore

$$\lim_{p\to\infty} p[1 - \rho(m^p; m)] = 0, \tag{4.16}$$

where

$$\rho(m^p; m) = \frac{ma(m^{p+1})}{a(m^p)}. \tag{4.17}$$

Now if we let $b(p) = m^p a(m^p)$ we have

$$\rho(m^p; m) = \frac{b(p+1)}{b(p)}. \tag{4.18}$$

Now $b(p+1)/b(p)$ is eventually positive since $\lim_{p\to\infty} \rho(m^p; m) = 1$.

Thus

$$\lim_{p\to\infty} p\left[1 - \frac{b(p+1)}{b(p)}\right] = 0 \tag{4.19}$$

and by Raabe's Test the series

$$\sum_{p=0}^{\infty} b(p) \tag{4.20}$$

diverges. But, for n sufficiently large,

$$|b(p)| = |m^p a(m^p)| \leq m^p M(m^p)^{-\alpha}$$
$$= M(m^{1-\alpha})^p. \tag{4.21}$$

However,

$$M \sum_{p=1}^{\infty} (m^{1-\alpha})^p \tag{4.22}$$

converges when m and $\alpha > 1$. Thus (4.20) converges absolutely. This is a contradiction and the theorem follows.

An interesting class of functions for which $T_{.m}$ will be useful is considered in the following theorem. For these functions one would expect $T_{.m}$ to be more effective than T_{+m}. This latter statement can be easily verified by investigating $R(n; m)$ and $\rho(n; m)$. That is, by noting that $\lim_{n\to\infty} R(n; m) = 1$ and $\lim_{n\to\infty} \rho(n; m) \neq 1, 0$ for the class of functions discussed in the following theorem.

THEOREM 4.4: *If* $a(n) = \dfrac{u(n)}{v(n)}$, *where* $u(n) = \sum_{i=0}^{N} C_i n^{r-i}$, $v(n) = \sum_{i=0}^{M} d_i n^{s-i}$, $r \geq N$, $S \geq M$, $S > r+1$ *and* $C_0, d_0 \neq 0$, *then* $T_{.m}[S(nm)]$ *converges to* S, i.e.,

$$\lim_{n\to\infty} T_{.m}[S(nm)] = \sum_{n=1}^{\infty} a(n).$$

PROOF:

$$\frac{ma(mn)}{a(n)} = \frac{m \sum_{i=0}^{N} C_i (nm)^{r-i}}{\sum_{i=0}^{M} d_i (nm)^{s-i}} \cdot \frac{\sum_{i=0}^{M} d_i n^{s-i}}{\sum_{i=0}^{N} C_i n^{r-i}}. \tag{4.23}$$

Thus

$$\lim_{n\to\infty} \rho(n; m) = m^{r-s+1} \neq 1 \tag{4.24}$$

and Corollary 4.1 applies.

THEOREM 4.5: *If* $\rho_m \neq 0, 1$, *then* $T_{.m}[S(mn)]$ *converges more rapidly than* $S(nm)$ *if and only if*

$$\lim_{n\to\infty} \frac{S - S(nm)}{S - S(n)} = \rho_m. \tag{4.25}$$

PROOF: The theorem is a special case of Theorem 2.2.

Before we proceed further let us recall that $T_{.m}$ was initially considered to circumvent the problem of $R_m = 1$ in T_{+m}. Consequently, a question which naturally arises and which is of some importance is the following: Is the class of sequences in Theorem 4.5 distinct from the class of sequences in Theorem 3.8? That is, we know that if $R_m \neq 0, 1$, then a necessary and sufficient condition for T_{+m} to induce more rapid convergence on a sequence is that

$$R_m = \lim_{n\to\infty} \left[\frac{(S - S(n+m))}{(S - S(n))} \right].$$

But by Theorem 4.5 $T_{.m}$ will also induce more rapid convergence if (4.25) holds. Thus, do there exist sequences for which both Theorem 4.5 and Theorem 3.8 apply and if so which transformation would be preferable? The dilemma is effectively resolved by the following theorem.

THEOREM 4.6: *If* $\lim R(n, 1) = R_1$, $0 < |R_1| < 1$ *and* ρ_m *exists, then* $\rho_m = 0$.

PROOF: Suppose $\rho_m \neq 0$, then we have

$$R_1 = \lim_{n\to\infty} \frac{a(nm+1)}{a(nm)} \tag{4.26}$$

$$= \lim_{n\to\infty} \left[\frac{a(nm+1)}{a(nm+2)} \cdot \frac{a(nm+2)}{a(nm+3)} \cdots \frac{a(mn+m-1)}{a(nm+m)} \cdot \frac{a(nm+m)}{a(n+1)} \cdot \frac{a(n+1)}{a(n)} \cdot \frac{a(n)}{a(nm)} \right]$$

$$= \frac{1}{R_1} \cdot \frac{1}{R_1} \cdot \cdots \cdot \frac{1}{R_1} \cdot \frac{\rho_m}{M} \cdot R_1 \cdot \frac{M}{\rho_m} = \frac{1}{R_1^{m-2}}. \tag{4.27}$$

Thus $R_1^{m-1} = 1$ and, since $m \geq 2$, $|R| = 1$. The theorem therefore follows. We see therefore that with the exception of $R_1 = -1$ the two theorems refer to disjoint sets of sequences. Note that in the event $m = 2k$ that $R_m = 1$ and in that case the two theorems refer to two distinct classes of sequences without exception. That is, if $T_{\cdot m}$ induces more rapid convergence by Theorem 4.5, then Theorem 3.8 does not apply since we must have $R_m = 1$ by Theorem 4.6.

LEMMA 4.2: *If* $R_1 = 0$, *then* $\rho_m = 0$.

PROOF: If $\epsilon > 0$, then there exists an integer N such that if $n \geq N |R(n; 1)| < 1$ and $|R(n; 1)| < \epsilon$. Thus $|a(n+1)| < |a(n)|$ and $|a(n+1)| < \epsilon |a(n)|$. Since $nm \geq n+1$, it follows that $|a(nm)| < |a(n+1)| \leq |a(n+1) < |a(n)|$. Hence, $\left|\frac{a(nm)}{a(n)}\right| < \epsilon$ and the lemma follows.

LEMMA 4.3: *If* $R_1 \neq -1$ *and* ρ_m *exists, then* $\frac{ma(nm+p)}{a(n+1)} \to \rho_m$ *as* $n \to \infty$ *for each integer* $p \geq 0$.

Moreover, if $R = -1$, *then* $\frac{ma(nm+p)}{a(n+1)} \to (-1)^{p+1}\rho_m$ *as* $n \to \infty$.

PROOF: If $|R_1| < 1$, then, by Theorem 4.6 and Lemma 4.2, $\rho_m = 0$. Also, as in Lemma 4.2, $|a(nm+p) < |a(n+2)| < \epsilon |a(n+1)|$. Thus $\frac{ma(nm+p)}{a(n+1)} \to 0 = \rho_m$. Consideration of the expression

$$\frac{a(nm+p)}{a(n+1)} = \frac{a(nm)}{a(n)} \cdot \frac{a(nm+1)}{a(nm)} \cdot \cdots \cdot \frac{a(nk+p)}{a(nk+p-1)} \cdot \frac{a(n)}{a(n+1)}$$

yields the required result for $|R_1| = 1$.

THEOREM 4.7: *If* $S(n)$ *is monotone,* $R(n, 1) \to R$ *and* ρ_m *exists, then* $\frac{S - S(mn)}{S - S(p)} \to \rho_m$ *as* $n \to \infty$.

PROOF: $S(n)$ being monotone implies that $R_1 \neq -1$. Also

$$\frac{S(mn) - S(mn-m)}{S(n) - S(n-1)} = \frac{a[(n-1)m+1]}{a(n)} + \cdots + \frac{a[(n-1)m+m]}{a(n)}.$$

By lemma 4.3, this converges to ρ_m. Thus, by Theorem 3.2, $\frac{S - S(mn)}{S - S(n)} \to \rho_m$.

THEOREM 4.8: *If* $S(n)$ *is monotone,* $R(n; 1)$ *converges and* $\rho_m \neq 0, 1$, *then* $T_{\cdot m}$ *converges more rapidly than* $S(nm)$.

PROOF: In this case $\frac{S - S(nm)}{S - S(n)} \to \rho_m$ and Theorem 4.5 applies.

THEOREM 4.9: *If* $a(n)$ *is of constant sign and satisfies the conditions of Theorem* 4.4, *then* $T_{\cdot m}$ *converges to* S *more rapidly than* $S(mn)$.

PROOF: $Rn; 1) \to 1$ and $\rho_m = m^{r-s+1} \neq 0, 1$. Thus Theorem 4.8 applies.

The following examples illustrate the advantage of $T_{\cdot m}$ over T_{+m} when $R_m = 1$.

EXAMPLE 4.1: Let $S(n) = \frac{n}{n+1}$. Then

(i) $S = 1$

(ii) $a(n) = \frac{1}{n(n+1)}$

(iii) $R_m = 1$ and $\rho_m = \frac{1}{m}$

(iv) $T_{+m}[S(n+m)]=\dfrac{2n+m}{2n+m+1}$

(v) $T_{\cdot m}[S(nm)]=1.$

Obviously, T_{+m} and $S(n+m)$ converge at the same rate while $T_{\cdot m}$ is exact and converges more rapidly than $S(nm)$ and $S(n+m)$.

EXAMPLE 4.2: The transforms considered are T_{+1} and $T_{\cdot 2}$.

Let

$$S=\frac{(1+1)+1}{1+1}+\frac{2^4+2^2+1}{2^2(2^4+1)}+\frac{3^4+3^2+1}{3^2(3^4+1)}+\cdots$$

Then $a_n=\dfrac{n^4+n^2+1}{n^2(n^4+1)}$ and $R_1=1$, $\rho_m=1/2$.

Since $a_n>0$, $\rho_m\neq 0$, 1 and $R(n;\ 1)$ converges, it follows that $T_{\cdot 2}$ converges more rapidly to S than $S(2n)$. However, this example is given by Lubkin [11] to show that T_{+1} diverges!

EXAMPLE 4.3: The transforms considered are T_{+1} and $T_{\cdot 2}$. Let $S_n=\sum_{k=1}^{n}\dfrac{1}{k^2}$. Then $S_n\to \Pi^2/6$.

The transforms were applied in the forms

$$T_{+1}[S(n+1)]=\frac{S(n+1)S(n-1)-[S(n)]^2}{S(n+1)+S(n-1)-2S(n)}$$

and

$$T_{\cdot 2}[S(2n)]=\frac{2S(n)S(2n-1)-S(n-1)S(2n)-S(n)S(2n)}{S(n)-S(n-1)-2[S(2n)-S(2n-1)]}$$

for computational ease.

$S(1)$ through $S(10)$ was the only information used. The transforms were then reapplied to the resulting sequences. The process was repeated to get the final results.

The process is illustrated in the following tabular form. The symbols T_{+1}^n and $T_{\cdot 2}^n$ refer to n applications of the respective transforms.

n	S_n	$T_{\cdot 2}[S(2n)]$	Error
1	1.000 000 0		
2	1.250 000 0		
3	1.361 111 1	0.159 722 2 D 01	0.477 118 4 D–01
4	1.423 611 1		
5	1.463 611 1	.162 166 7 D 01	.232 674 0 D–01
6	1.491 388 3		
7	1.511 796 9	.163 123 3 D 01	.137 010 7 D–01
8	1.527 421 9		
9	1.539 767 5	.163 592 4 D 01	.900 971 6 D–02
10	1.549 767 5		

$T_{\cdot 2}^2[S(2n)]$	Error
0.164 480 6 D 01	0.127 879 3 D–03

The notation D 01 above indicates the position of the decimal. Thus 0.1278793 D–03 is 0.0001278793.

n	S_n	$T_{+1}[S(n+1)]$	Error
1	1.0000000		
2	1.2500000	0.145 000 0 D 01	0.194 934 1 D−00
3	1.3611111	.150 396 8 D 01	.140 965 8 D−00
4	1.4236111	.153 472 2 D 01	.110 211 8 D−00
5	1.4636111	.155 452 0 D 01	.904 138 6 D−01
6	1.4913888	.156 831 2 D 01	.776 221 0 D−01
7	1.5117969	.157 846 4 D 01	.664 703 5 D−01
8	1.5274219	.158 624 6 D 01	.586 884 9 D−01
9	1.5397675	.159 239 9 D 01	.525 347 6 D−01
10	1.5497675		

$T^2_{+1}[S(n+1)]$	Error
1.157 546 465 879 801 D 01	0.694 694 080 502 174 D−01
0.159 029 607 068 481 D 01	.546 379 961 634 098 D−01
.159 998 128 111 844 D 01	.449 527 857 297 853 D−01
.160 677 629 679 179 D 01	.381 577 700 564 282 D−01
.161 179 840 516 212 D 01	.331 356 616 861 027 D−01
.161 565 806 547 658 D 01	.292 760 013 716 415 D−01

$T^3_{+1}[S(n+1)]$	Error
0.161 820 895 875 573 D 01	0.267 251 080 924 982 D−01
.162 275 177 312 803 D 01	.221 822 037 201 894 D−01
.162 602 451 412 998 D 01	.189 095 527 182 483 D−01
.162 847 324 260 966 D 01	.164 608 242 385 657 D−01

$T^3_{+1}[S)n+1)]$	Error
0.163 445 775 421 854 D 01	0.104 763 126 296 867 D−01
.163 575 016 014 021 D 01	.918 390 670 801 172 D−02

In the above it should be noted that two applications of $T_{.2}$ have yielded better results than four applications of T_{+1} even though each application of T_{+1} gave a better result than the previous application of T_{+1}.

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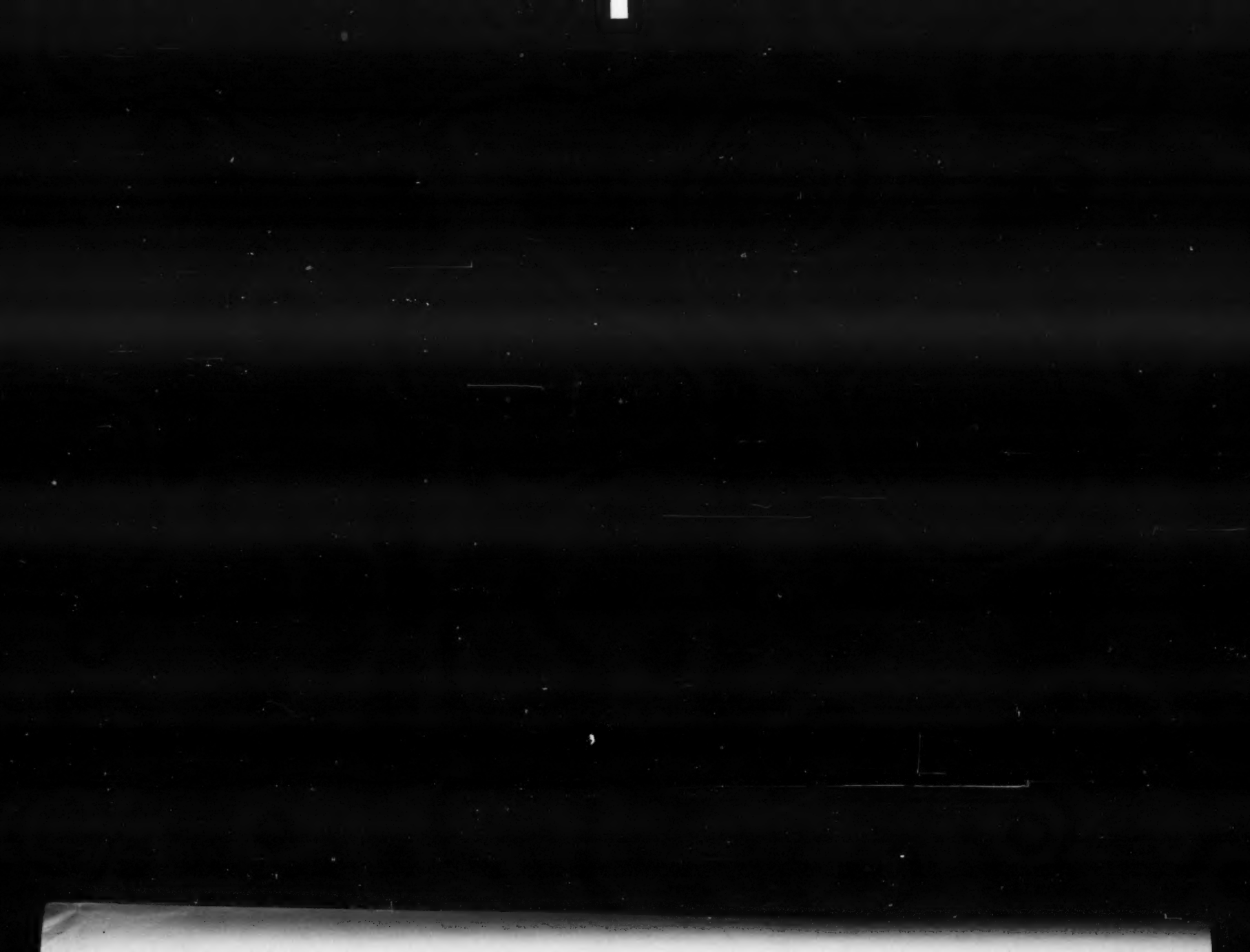
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